Rocket Trajectory Analysis

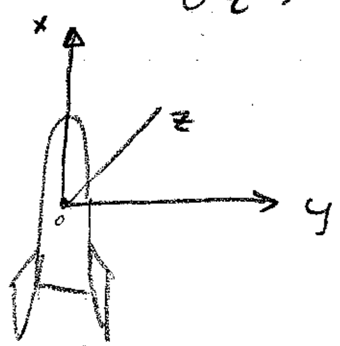
- how high will the rocket go?
- where will it go?
- what affects the trajectory?
- how do we model this mathematically?
- how do we solve the eqns?
- how do we compare to our exp. data?

Cons of Momentum - rigid body, axes fixed to rocket

axes: OX : local vertical (up)

OY : local east

OZ : local north



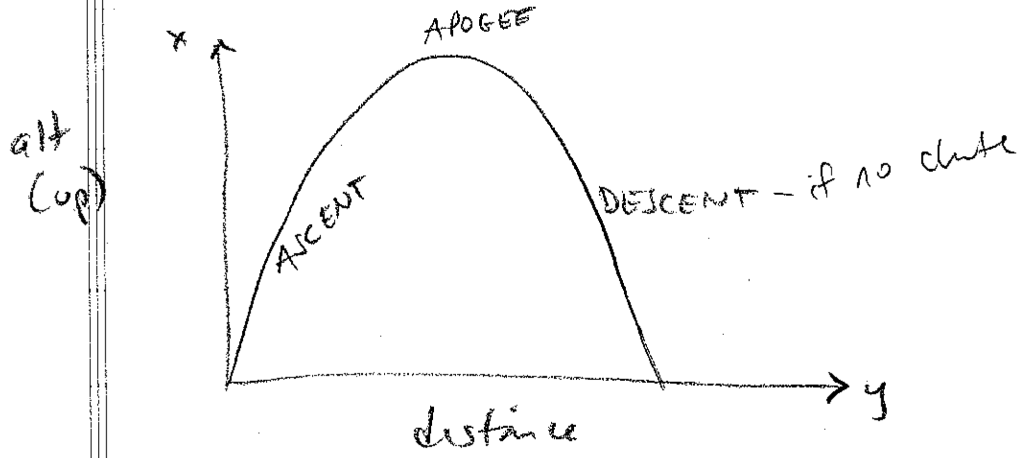
(DOF = degree of freedom)

We could do full 6DOF, but 3DOF will likely be adequate for the type of trajectory we ideally want.

What will affect our rocket's flight?

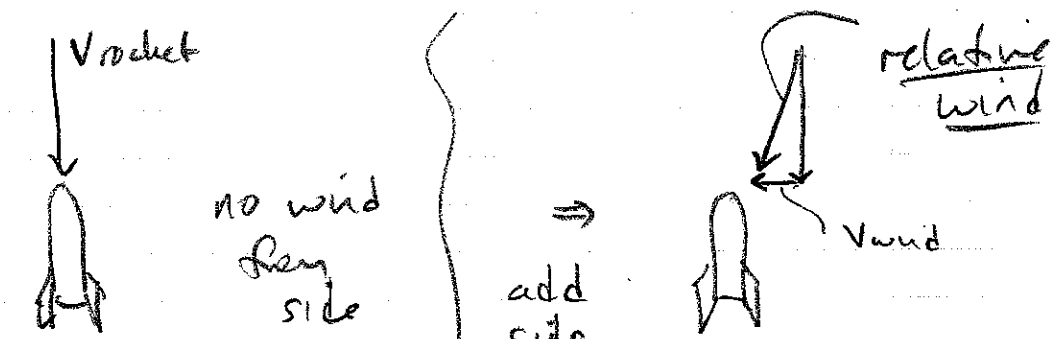
- gravity
- thrust from motor
- side wind
- aerodynamic forces

We want to predict something like this:



Why did we draw it this way? Why did I indicate the rocket "turning" (not going straight up?) What is wind direction in this schematic?

- <u>weather cocking</u> – if there is a side wind, the rocket can turn into the wind. Why?



⇒ aerodyn force is DRAG

⇒ still get DRAG, but the rocket now sees the incoming air at an <u>angle of attack</u>

⇒ LIFT generated

- Where does this new aerodyn force act?

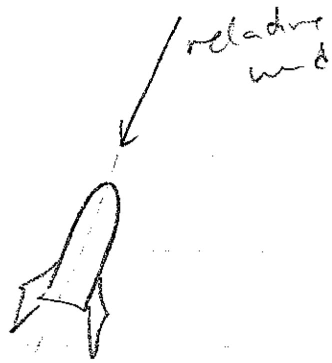
  ⇒ <u>center of pressure</u>. (CP)

  - Is it the same as the center of mass for our rocket? ⇒ <u>NO</u>

⇒ a moment is generated about the CG and the <u>rocket rotates</u> (↳ center of gravity)

- When does it stop rotating?

  It stops when it is aligned in the direction of the relative wind (when the angle of attack is zero)



So the Lift force (and that moment) is a function of the angle between the relative wind and the rocket direction

⇒ so we know an angular momentum eqn will be needed to predict this ⇒ about z axis

⇒ we also need two linear moment eq'ns to get X, Y ⇒ x-dir moment
y-dir moment.

In the body frame (simplified for our 3DOF case and small changes in angle)

$$M \frac{dV_{cx}}{dt} = G_x + T_x + X \qquad \text{x-dir moment}$$

$$M \frac{dV_{cy}}{dt} = G_y + Y \qquad \text{y-dir moment}$$

$$J_z \frac{d\omega_z}{dt} = M_{Az} \qquad \text{z-dir ang. moment}$$

where V_{cx}, V_{cy} = velocity of rocket in the body frame

G_x, G_y = components of gravity force

T_x = thrust force

X, Y = components of aerodyn force

M_{Az} = aerodyn moment

What is the thrust term? From Prof [illegible]'s lecture:

$$T_x = \underbrace{\dot{m} V_r}_{\text{thrust curves}} + A_{exit}(P_{exit} - P)$$

→ data you took / will take

where $\dot{m}$ = mass flow of motor exhaust

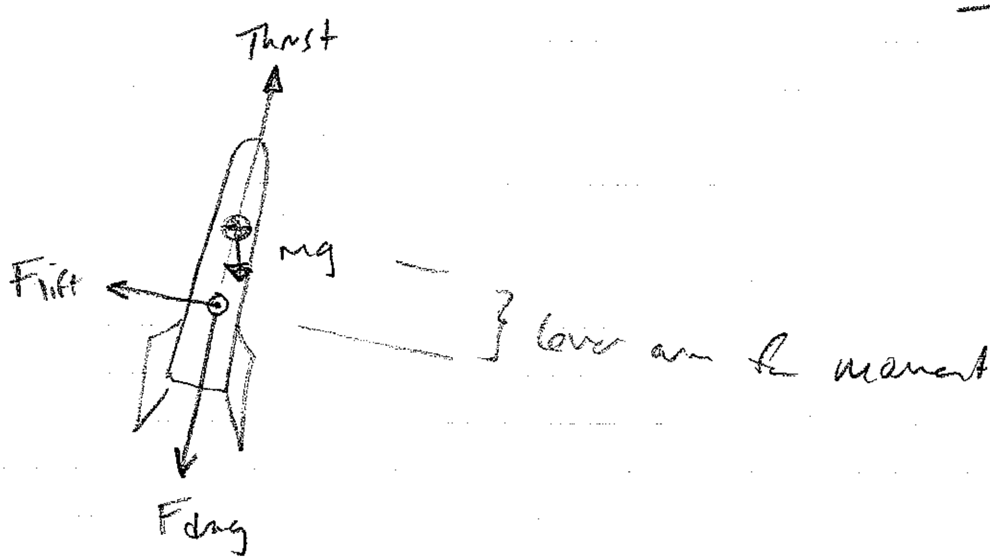
V_r = velocity of exhaust

P_{exit} = pressure of exiting exhaust

P = atm pressure

→ NOTE: unlike subsonic gas flows, which cannot physically support a ΔP across a free jet, supersonic flows CAN ⇒ shock waves, expansion fans.

Take ENGR 140 (Compressible flow)



All ok for ascent. Thrust curve as a f(time)

C_D ⇒ drag force

C_L → lift force + moment

gravity ⇒ gravity force

P → std atm.

What happens @ apogee?

Chute pops ⇒ add drag of chute (literature values OK)

$F_{drag} = \frac{1}{2} \rho C_D |V_{cx}^2| V_{cx}$

- I'll let you figure out how to set up gravity force components, aerodyn moments, relative wind angle.

How to solve?

recall Taylor series:

$$V_{cx}(t+\Delta t) = V_{cx}(t) + \frac{dV_{cx}}{dt}\Delta t + \frac{d^2V_{cx}}{dt^2}\frac{\Delta t^2}{2!} + \ldots$$

ignoring higher order terms (HOT)

$$\Rightarrow \frac{dV_{cx}}{dt} = \frac{V_{cx}(t+\Delta t) - V_{cx}(t)}{\Delta t}$$

so $$m\frac{dV_{cx}}{dt} = G_x + T_x + X$$

$$\Rightarrow m\left[\frac{V_{cx}(t+\Delta t) - V_{cx}(t)}{\Delta t}\right] = G_x + T_x + X$$

everything known @ time $t=0$

$$V_{cx}(t+\Delta t) = \underbrace{\frac{\Delta t}{M}\left[G_x + T_x + X\right]\Big|_{\text{at time } t} + V_{cx}(t)}_{\text{all known}}$$

so can calculate $V_{cx}(t+\Delta t) \Rightarrow$ march forward in time recall from Prof Wang's lecture that we can go from body axes to global axes.

For small rotation angles, 3 DOF

$$V_x = v_{cx} - v_{cy}\,\phi_z$$

$$V_y = v_{cx}\,\phi_z + v_{cy}$$

global velocities

(How does ϕ_z relate to our eqns? recall $\dot{\phi}_z = \omega_z$)

From IMU lecture

$$\underbrace{\begin{pmatrix} U_x \\ U_y \\ U_z \end{pmatrix}}_{\text{global}} = \underbrace{\begin{bmatrix} 1 & -\phi_z & \phi_y \\ \phi_z & 1 & -\phi_x \\ -\phi_y & \phi_x & 1 \end{bmatrix}}_{\substack{\text{rotation matrix } R \\ \text{for small } \phi\text{'s}}} \underbrace{\begin{pmatrix} v_{cx} \\ v_{cy} \\ v_{cz} \end{pmatrix}}_{\text{local}}$$

How do we get ϕ's? Also, the rocket continually changes position/orientation. How do we handle that?

We need to get R matrix from some variables we know!

Consider

$$\bar{u}(t) = \bar{R}(t)\,\bar{v}(t) \quad \text{at time } t.$$

At time $t + \delta t$, we can write

$$\underbrace{\bar{S}(t)}_{\substack{\text{a rotational}\\ \text{matrix}}}\,\bar{v}(t+\delta t) = \bar{v}(t)$$

So $$\bar{u}(t) = \underbrace{\bar{R}(t)\,\bar{S}(t)}_{\bar{R}(t+\delta t)}\,\bar{v}(t+\delta t)$$

$S(t)$ is a rotation matrix like $R(t)$, with small rotations $\delta\phi_x, \delta\phi_y, \delta\phi_z$ that occur during δt

$$S(t) = \begin{bmatrix} 1 & -\delta\phi_z & \delta\phi_y \\ \delta\phi_z & 1 & -\delta\phi_x \\ -\delta\phi_y & \delta\phi_x & 1 \end{bmatrix} = \underline{I} + \delta\underline{\Phi}$$

So? Well, we can rewrite this as a derivative of R

$$\dot{R}(t) = \lim_{\delta t \to 0}\left[\frac{R(t+\delta t) - R(t)}{\delta t}\right]$$

$$= \lim_{\delta t \to 0}\left[\frac{R(t)\,S(t) - R(t)}{\delta t}\right] = \lim_{\delta t \to 0}\left[\frac{R(t)\overbrace{(\underline{I} + \delta\underline{\Phi})}^{S(t)} - R(t)}{\delta t}\right]$$

$$= R(t)\lim_{\delta t \to 0}\frac{\delta\underline{\Phi}}{\delta t} = R(t)\,\dot{\underline{\Phi}}$$

Recall that $\dot{\phi}_x = \omega_x$; $\dot{\phi}_y = \omega_y$; $\dot{\phi}_z = \omega_z$

so we can define

$$\Omega = \frac{d}{dt}\Phi = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

$[\dot{R}(t) = R(t)\dot{\Phi}]$

so $\dot{R}(t) = R(t)\,\Omega(t)$ $\quad \left[\frac{dR}{dt} = R\Omega \,;\ \frac{dR}{R} = \Omega\, dt \right]$

can solve this $\Rightarrow$ $R(t) = R(0)\exp\left[\int_0^t \Omega(t)\,dt\right]$

What does this mean?

We integrate the angular velocities $(\omega_x, \omega_y, \omega_z)$

for small δt, $\displaystyle\int_t^{t+\delta t} \Omega\, dt \approx \Omega(t)\,\delta t = B$

so $R(t+\delta t) = R(t)\exp B$

expand exp $\rightarrow$ see Wang's IMU lecture.

w/ $\sigma \equiv |\omega_b\, \delta t|$

$$\boxed{R(t+\delta t) = R(t)\left(I + \frac{\sin\sigma}{\sigma}B + \frac{1-\cos\sigma}{\sigma^2}B^2\right)}$$

update as each new sample is available

where $B = \begin{bmatrix} 0 & -\omega_z\delta t & \omega_y \delta t \\ \omega_z \delta t & 0 & -\omega_x\delta t \\ -\omega_y \delta t & \omega_x \delta t & 0 \end{bmatrix}$

remember, we use this to relate the local coordinate system to the global. Each time we get new ω's, we update the rotation matrix

ERRORS in IMU measurement

- OBJECTIVE – compare your model prediction to what actually happened during launch (exp data)

⇒ IMU measurements can have noise and bias

- constant bias errors: non-zero output when there is no accel or rotation
- moving bias errors: e.g. temperature effects can move the bias

Integration of these measurements can produce large errors in position, velocity, and heading calculations.

- can do error modeling to estimate the std dev of the estimate error over time
- recursive filtering (Kalman)
  ↳ see Wang's notes

APPENDIX

Full 6DOF Eqns — Body Frame

$$m\left(\frac{dv_{cx}}{dt} + \omega_y v_{cz} - \omega_z v_{cy}\right) = G_x + T_x + X + F_{kx}$$

$$m\left(\frac{dv_{cy}}{dt} + \omega_z v_{cx} - \omega_x v_{cz}\right) = G_y + T_y + Y + F_{ky}$$

$$m\left(\frac{dv_{cz}}{dt} + \omega_x v_{cy} - \omega_y v_{cx}\right) = G_z + T_z + Z + F_{kz}$$

$$J_x \frac{d\omega_x}{dt} + (J_z - J_y)\,\omega_y \omega_z = M_{px} + M_{Ax} + M_{kx}$$

$$J_y \frac{d\omega_y}{dt} + (J_x - J_z)\,\omega_z \omega_x = M_{py} + M_{Ay} + M_{ky}$$

$$J_z \frac{d\omega_z}{dt} + (J_y - J_x)\,\omega_x \omega_y = M_{pz} + M_{Az} + M_{kz}$$

F_{kx}, F_{ky}, F_{kz} = Coriolis Force Components

X, Y, Z = components of aero force

T_x, T_y, T_z = components of thrust force

G_x, G_y, G_z = " " gravity "

M_{px}, M_{py}, M_{pz} = components of thrust moment

M_{Ax}, M_{Ay}, M_{Az} = " " aero "

M_{kx}, M_{ky}, M_{kz} = " " Coriolis "