LECTURE 8

Modal Vibration

* Basic Ideas:

Modal Analysis : Charcterization of vibrational mode shapes and corresponding frequencies of a physical system.

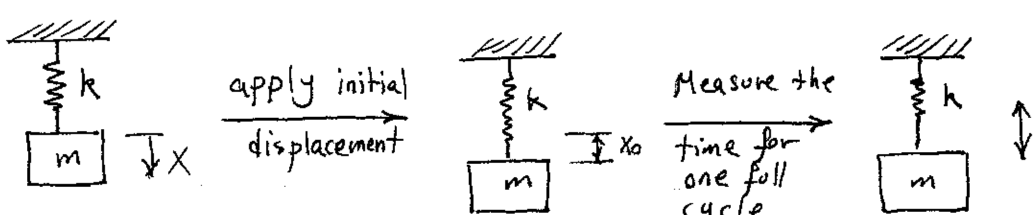
Properties of a system : Natural frequencies and modal shapes

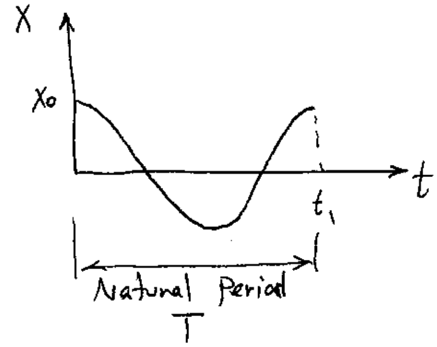
Dynamic vs. Static : Varying in time vs. constant in time

Dynamic loads : Loads applied dynamically (over time, varying)

Natural period of vibration of a structure :

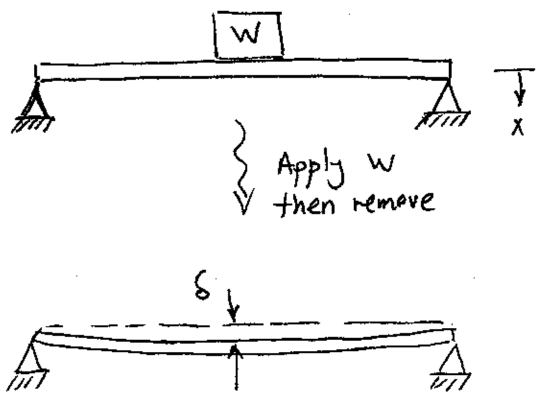
Time required for the structure to undergo one complete cycle of free vibration.

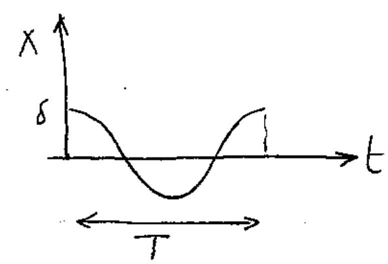




Or can call it natural frequency $\omega = \dfrac{2\pi}{T}$

Same for a structural component ...





If time variation of the load application is such that it is smaller or comparable to natural period of vibration of the structure, then your structure has no time to undergo a free vibration cycle and start to respond to the dynamic load.

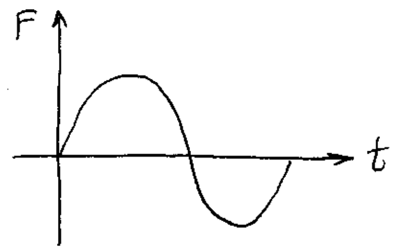
→ The response associated to this type of loading is the dynamic response of the structure.

→ Structures undergo through more severe stresses and strains when dynamic loads are applied.

Sources of Dynamic Loads

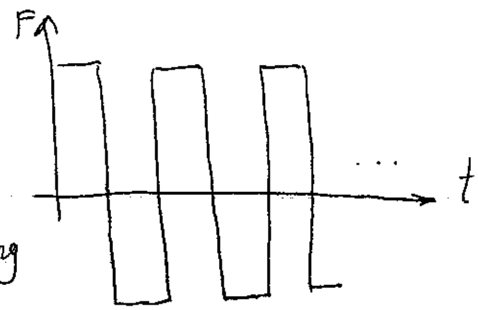
Various sources of dynamic loads can be named.

1) Simple Harmonic



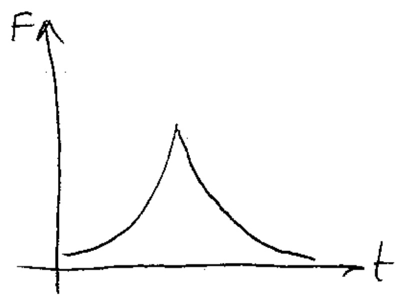
Example: Interaction of ocean waves with a structure.

2) Periodic, non harmonic



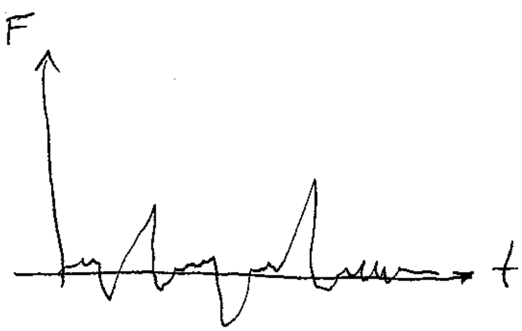
Example: Loading and unloading a structure

3) Non-Periodic, short duration



Example: sudden impact to a structure.

4) Non-periodic, long duration



Example: Bridge with ongoing traffic

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An important observation in dynamic loading is the fact that inertia plays an important role. . .

* Analysis of Dynamic Loading:

How do you approach the problem of analyzing a structure with dynamic loading?

a) Model your physical problem

- Geometry
- Kinematics
- Material
- Loading

b) Derive governing equations (mostly differential equations)

c) Solve the equations

d) Interpret the results and refine and repeat!

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What else can you do to characterize a structure?

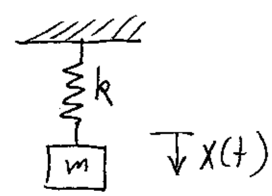
Measurements! $\Rightarrow$ Experimental studies to validate a model or help develop a model.

* A Simple model

Let's look at a very simple model that makes the building block of a lot of structural dynamics analysis tools ...

Free, undamped vibration of a spring-mass system

This is a single degree of freedom SDOF system. m : mass k : spring constant



We saw in E59 that this physical model has a differential equation associated to it:

$$m\ddot{x} + kx = 0 \qquad (1)$$

with initial conditions of $x(t=0) = x_0$ and $\dot{x}(t=0) = \dot{x}_0$

we recall from E59 that solution of (1) is

$$x(t) = \overset{\text{amplitude}}{X} \sin(\overset{\text{frequency}}{\omega} t + \overset{\text{phase}}{\phi}) \qquad (2)$$

with $X = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega}\right)^2}$ and $\phi = \tan^{-1}\left(\frac{x_0 \omega}{\dot{x}_0}\right)$

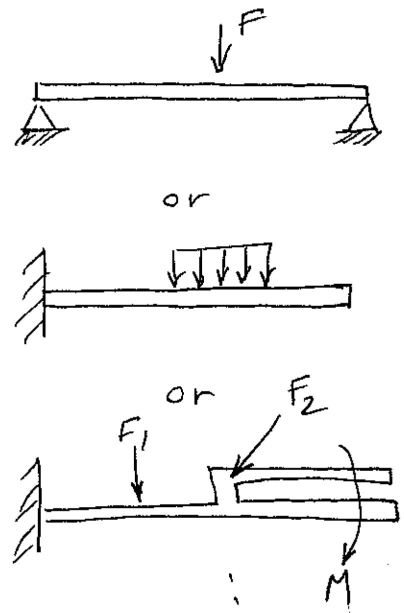
and $\omega = \sqrt{k/m}$.

So in a nut shell, if we manage to extend this idea to a structural element, say a beam then we should be able to see how this structural element or even the structure as a whole responds to a free or forced (loaded) vibration (dynamic loading) case.

*Beams

Beams are one of the most important components in structural engineering. Examples of beams are bridges, walkways, rockets(?) and ... .

Simple representation of beams is shown below:



Some important properties and characteristics of a beam are:

1) Cross section  ...

2) Length and supports   Fixed - Free, simply supported, ...

3) Loading   concentrated, distributed, dynamic, static, ...

4) Material   steel, wood, plastic, ...

Each of the above defines the model and parameters needed to analyze the beam.

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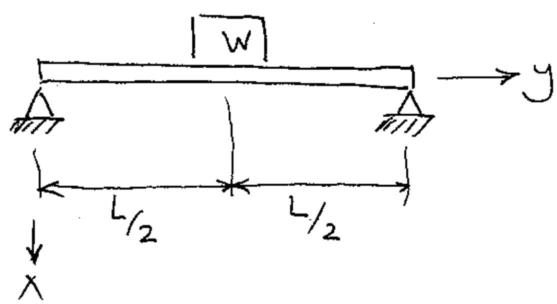
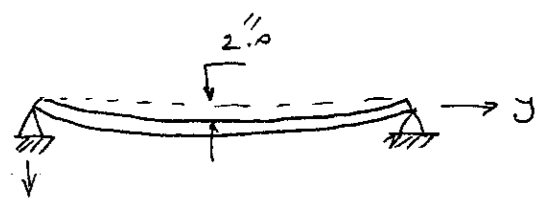
Let's study a beam problem that resembles our previous mass-spring free vibration problem.



Consider the beam shown. Let's say the natural frequency of the beam is given to us as 27.88 rad/s. → $\left(\frac{27.88}{2\pi} = 4.44 \text{ Hz}\right)$

As a result of applying load W, this beam deflects 2.0″



If load W is released at -3.0 in/sec, we want to find out the amplitude of the oscillation for the beam.

If we model our beam as a mass-spring system then

$$X(t) = X \sin(\omega t + \phi) \quad , \quad \omega = 27.88 \text{ rad/sec}$$

$$X_0 = 2.0 \text{ in.}$$

$$\dot{X}_0 = -3.0 \text{ in/sec}$$

$$X = \sqrt{X_0^2 + \left(\frac{\dot{X}_0}{\omega}\right)^2} = \sqrt{2.0^2 + \left(\frac{3.0}{27.88}\right)^2} = 2.005''$$

$$\dot{X}(t) = X\omega\cos(\omega t + \phi) \xrightarrow{\dot{X}_{max}} \dot{X}_{max} = X_{max}\,\omega = 2.005 \times 27.88 = 55.9 \text{ in/sec}$$

$$\ddot{X}(t) = -X\omega^2 \sin(\omega t + \phi) \xrightarrow{\ddot{X}_{max}} \ddot{X}_{max} = X\omega^2 = 1558.48 \text{ in/sec}^2$$

$$\phi = \tan^{-1}\left(\frac{X_0\,\omega}{\dot{X}_0}\right) = \tan^{-1}\left[\frac{2 \times 27.88}{-3}\right] = 273^\circ$$

Having $X, \dot{X}, \ddot{X}$ and ϕ one can predict how this beam oscillates and what magnitudes of displacement, velocity and ... to expect. These can help design a component within the design requirements and limitations.

* Equivalent stiffness

One important observation is ω of the system which was known at the beginning.

We know that $\omega = \sqrt{\frac{k}{m}}$ so what is k for a beam?

we call k the "equivalent stiffness" of the beam.

k_{eq} depends on several factors:

Cross section $\xrightarrow{say}$ I

Length $\longrightarrow$ L

Elasticity $\longrightarrow$ E

For example discussed

$$k_{eq} = \frac{192\,EI}{L^3}$$

The important question, now that we saw the analogy between mass-spring and beams is how to model a dynamic load response of a beam?

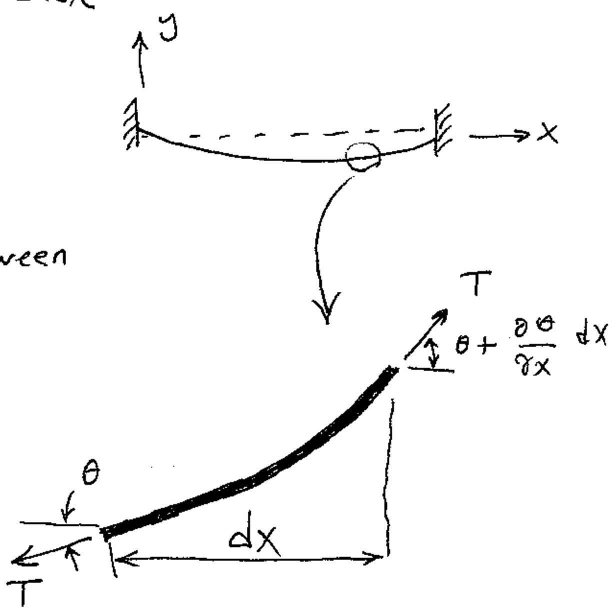
In other words, how do you derive appropriate differential equations?

Let's look at a problem that pretty much resembles what you will be testing in the lab. We hope to extend the results obtained for this model to that of a rocket structure (hollow cylinder).

* Transverse vibration of a pretensioned cable (e.g. Guitar string)

Consider a uniform, elastic cable with mass per unit length γ.

Let's apply a tension T between the two fixed points.



For small deflections use $\vec{F} = m\vec{a}$

$$\underbrace{T \sin\left(\theta + \frac{\partial \theta}{\partial x} dx\right)}_{\text{Force from the right side}} - \underbrace{T \sin \theta}_{\text{Force from the left side}} = \underbrace{\gamma dx}_{\text{mass}} \underbrace{\frac{\partial^2 y}{\partial t^2}}_{\text{acceleration}}$$

$$\xrightarrow{\text{simplify}} \quad \frac{\partial \theta}{\partial x} = \frac{\gamma}{T} \frac{\partial^2 y}{\partial t^2}$$

since $\theta = \frac{\partial y}{\partial x}$ is the slope of the cable,

$$\Rightarrow \quad \frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \quad \text{with} \quad c = \sqrt{\frac{T}{\gamma}} \qquad (*)$$

where c is the velocity of the wave propagation along the cable.

Solving equation $(*)$ requires the use of method of separation of variables which is a classical mathematics topic. Obtaining the solution results in

$$y(x,t) = \left(C_1 \sin\frac{\omega}{c}x + C_2 \cos\frac{\omega}{c}x\right)\left(A \sin\omega t + B\cos\omega t\right)$$

Use boundary conditions of $y(0,t) = y(L,t) = 0$

to get $C_2 = 0$

$$\Rightarrow y(x,t) = (A\sin\omega t + B\cos\omega t)\, C_1 \sin\frac{\omega}{c}x$$

and for $x = L \longrightarrow y(0,t) = 0 \Rightarrow C_1 \sin\frac{\omega L}{c} = 0$

if $C_1 = 0 \longrightarrow$ trivial solution, no meaningful result

$\Rightarrow \sin\frac{\omega L}{c} = 0$ is considered the correct choice.

$$\Rightarrow \frac{\omega L}{c} = n\pi \quad , \quad n = 1, 2, 3, \dots, \infty$$

$$\Rightarrow \omega_n = \frac{n\pi c}{L} \quad , \quad c = \sqrt{\frac{T}{\gamma}} \Rightarrow \omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\gamma}} \quad n = 1, 3, \dots, \infty$$

Each ω_n corresponds to a normal mode vibration having a mode shape y_n with a sinusoidal distribution

$$y_n = \sin \frac{n\pi x}{L}$$

add all $\Rightarrow$ solutions

$$y(x,t) = \sum_{n=1}^{\infty} (A_n \sin \omega_n t + B_n \cos \omega_n t) \sin \frac{\omega_n}{c} x$$

where $A_n = A C_1$ and $B_n = B C_1$

Now with initial conditions, $y(x,0) = Y_0 \sin \frac{\pi}{L} x$

initial shape

and zero velocity; $\frac{\partial y(x,0)}{\partial t} = 0$

$$\Rightarrow A_n = 0 \quad \text{and} \quad B_1 = Y_0 \; , \quad B_n = 0 \text{ for } n \neq 1$$

$$\Rightarrow \boxed{y(x,t) = \sum_{n=1}^{\infty} Y_0 \cos \omega_n t \sin \frac{n\pi}{L} x}$$

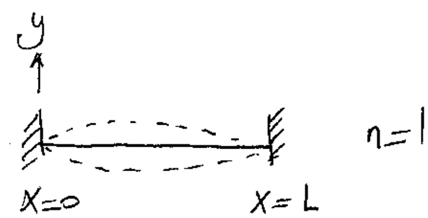
for all n modes

Displacement of the string as a function of space & time

What this equation tells us is that our string can oscillate with shapes formed by adding infinite numbers of individual modes.

Say $n=1$ only

$$y(x,t) = Y_o \cos \omega_1 t \sin \frac{\pi}{L} x$$

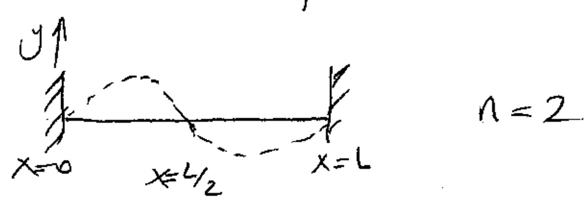


If $n=2$

$$y(x,t) = Y_o \cos \omega_1 t \sin \frac{\pi}{L} x + Y_o \cos \omega_2 t \sin \frac{2\pi}{L} x$$

Now with ω_2 and $\sin \frac{2\pi}{L} x$, 3 locations yield $y=0$:

$X=0$, $X=L$ and $x=\frac{L}{2}$ so mode 2 shape should look like this.

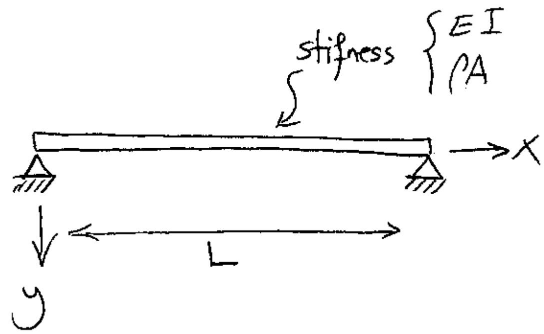


while mode $n=1$ is the same as before, mode $n=2$ exhibits a different shape mode.

Finally let's extend the idea to a transverse vibration of uniform beams ...

This time replace the string with a beam shown below:

(This can be your rocket !!)



EI is a measure of how stiff the beam is and its area

ρA is for inertia and material

"[redacted] beam"

we skip the mathematical details of the solution and write the expression:

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} (A_n \sin \omega_n t + B_n \cos \omega_n t) \sin \frac{n\pi x}{L}}$$

→ Recall the string problem on page 13 !

Our goal in the lab is to find ω_n's

$$\omega_n = (\beta_n L)^2 \sqrt{\frac{EI}{\rho A L^4}}$$

note EI and ρA and L playing major role in frequency of oscillation !

See other notes for free-free solution!

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where β_n for [redacted] beam is $\frac{n\pi}{L}$

* Summary :

We will be using tap test as excitation of the beam (Rocket body) and will measure response of the beam (with accelonameters mounted on the rocket) in <u>time</u> and <u>frequency</u> domains.

By obtaining all response frequencies (ω_n's) we can find out which modes ($y(x,t)$'s) have been excited as the result of our test. Knowing theoretical values of ω_n enables us to compare the experimental results with analytical results.

example of normal modes for the bean :

$$\omega_2 = \left(\frac{2\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}}$$

$n=2$

$$\omega_3 = \left(\frac{3\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}}$$

$n=3$

* Goals for experiments

* You will be using LabVIEW, DAQ and RDAS systems to acquire data and present them in different domains (time, frequency)
* You will be given basic dimensions, properties and other necessary physical parameters to compute the theoretical frequencies of the free-free beam test.