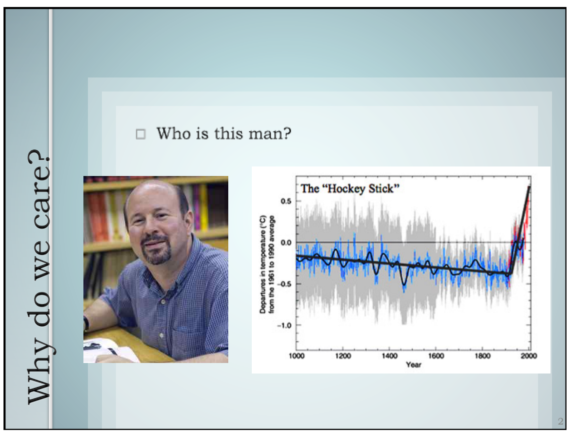
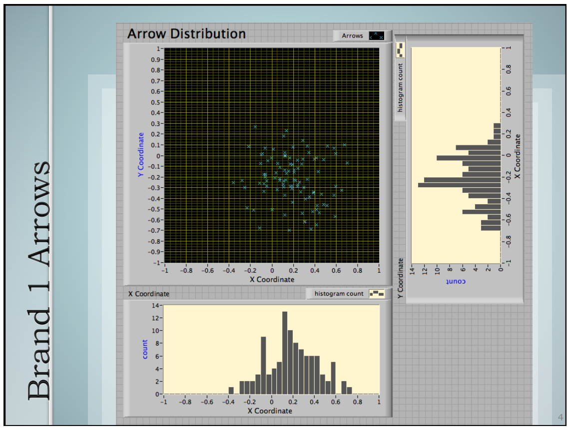
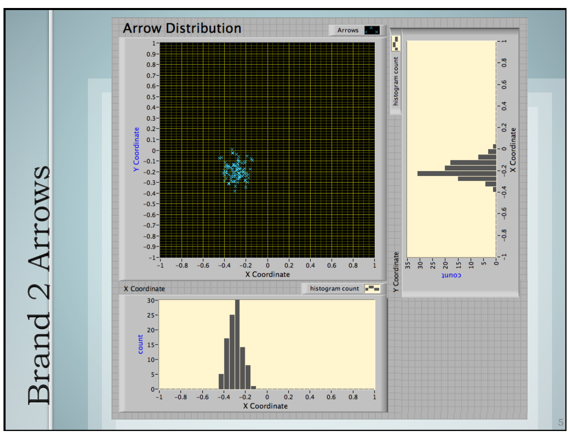

Measuring Things
E80 Spring 2015


Why do we care?
Who is this man?
The "Hockey Stick"
Year


Your Mission
Mythbuster Tech Advisor
Testing Arrows for a crossbow
MYTHBUSTERS


Brand 1 Arrows
Arrow Distribution
Arrows
Y Coordinate
X Coordinate
histogram count
count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Arrow Distribution
Arrows
Y Coordinate
X Coordinate
histogram count
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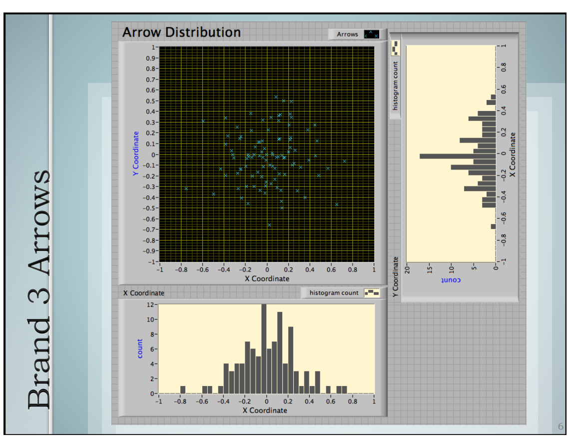
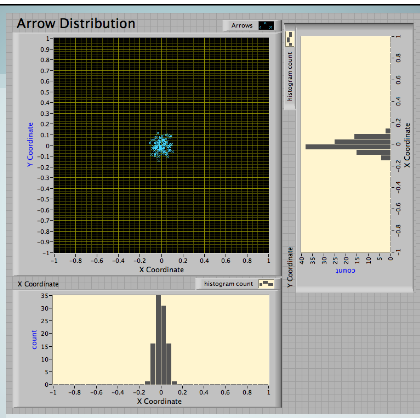

Brand 3 Arrows
Arrow Distribution
Arrows
Y Coordinate
X Coordinate
histogram count
count

Brand 4 Arrows



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Crossbow Questions

- Where is the center of the distribution, i.e., what is the aiming point?
- How certain are you?
- If we aim the crossbow at the apple, how likely is Buster™ to get an arrow in the head?

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Measured Value

- Where is the center of the distribution, i.e., where should we put the target?
- Use the *sample mean.*

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

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Error Estimate

- How certain are you? $\pm\lambda$
- Use *Estimated Standard Error* and Student's *t*-test

$$\lambda = tESE = \frac{tS}{\sqrt{N}}$$

$$S = \sqrt{S^2}$$

$$S^2 \equiv \frac{1}{N-1}\sum_{i=1}^{N} e_i^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \bar{x}\right)^2$$

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Students *t*-test

SIGNIFICANCE LEVEL FOR TWO-TAILED TEST						
df	.20	.10	.05	.02	.01	.001
1	3.078	6.314	12.706	31.821	63.657	636.619
2	1.886	2.920	4.303	6.965	9.925	31.598
3	1.638	2.353	3.182	4.541	5.841	12.941
4	1.533	2.132	2.776	3.747	4.604	8.610
5	1.476	2.015	2.571	3.365	4.032	6.859
10	1.372	1.812	2.228	2.764	3.169	4.587
20	1.325	1.725	2.086	2.528	2.845	3.850
30	1.310	1.697	2.042	2.457	2.750	3.646
40	1.303	1.684	2.021	2.423	2.704	3.551
60	1.296	1.671	2.000	2.390	2.660	3.460
120	1.289	1.658	1.980	2.358	2.617	3.373
∞	1.282	1.645	1.960	2.326	2.576	3.291

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Spread of Data

- If we aim the crossbow at the apple, how likely is Buster™ to get an arrow in the head?
- Use *sample standard deviation*

$$S = \sqrt{S^2}$$

$$S^2 \equiv \frac{1}{N-1}\sum_{i=1}^{N} e_i^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \bar{x}\right)^2$$

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Linear Fits

- How about fitting lines to sets of data?

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

- Use *linear least squares* fit.

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{N} (x_i - \bar{x})^2}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

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How Certain Are We?

- How good are the betas?

$$S_e = \sqrt{\frac{SSE}{N-2}} = \sqrt{\frac{\sum_{i=1}^{N} e_i^2}{N-2}} \qquad df = N-2$$

$$S_{\beta_0} = S_e \sqrt{\frac{1}{N} + \frac{\bar{x}^2}{\sum_{i=1}^{N} (x_i - \bar{x})^2}} \qquad \lambda_{\beta_0} = t S_{\beta_0}$$

$$S_{\beta_1} = S_e \sqrt{\frac{1}{\sum_{i=1}^{N} (x_i - \bar{x})^2}} \qquad \lambda_{\beta_1} = t S_{\beta_1}$$

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Why? Because!

- How certain is the calculated value of y for a chosen x?

$$S_y = S_e \sqrt{\frac{1}{N} + \frac{(x - \bar{x})^2}{\sum_{i=1}^{N} (x_i - \bar{x})^2}} \qquad \lambda_y = t S_y$$

- If we experimentally set x, how will the experimental y's spread (The arrow in the head question)?
- Use S_e like you previously used S.

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What About Functions?

- What if you want T?

$$T = \frac{1}{\frac{1}{T_0} + \frac{1}{\beta}\ln\frac{R}{R_0}}$$

- and you measure β and R?

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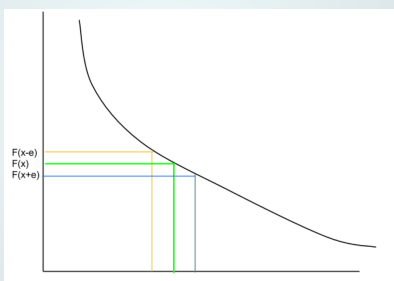
How error in x affects $F(x)$



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Error Propagation

- For $F = F(x, y, z, \cdots)$

$$F - F_{true} = \frac{\partial F}{\partial x}\left(x - x_{true}\right) + \frac{\partial F}{\partial y}\left(y - y_{true}\right) + \frac{\partial F}{\partial z}\left(z - z_{true}\right) + \cdots$$

- Let $\varepsilon_x = x - x_{true}$

$$\varepsilon_F = \frac{\partial F}{\partial x}\varepsilon_x + \frac{\partial F}{\partial y}\varepsilon_y + \frac{\partial F}{\partial z}\varepsilon_z + \cdots$$

$$\varepsilon_F = \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 \varepsilon_x^2 + \left(\frac{\partial F}{\partial y}\right)^2 \varepsilon_y^2 + \left(\frac{\partial F}{\partial z}\right)^2 \varepsilon_z^2 + \cdots}$$

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Thermistor Example

$$T = \frac{1}{\frac{1}{T_0} + \frac{1}{\beta}\ln\frac{R}{R_0}}$$

$$dT = \frac{\left[\left(\frac{1}{\beta^2}\ln\frac{R}{R_0}\right)d\beta - \frac{1}{\beta R}dR\right]}{\left[\frac{1}{T_0} + \frac{1}{\beta}\ln\frac{R}{R_0}\right]^2}$$

$$dT = T^2\left[\left(\frac{1}{\beta^2}\ln\frac{R}{R_0}\right)d\beta - \frac{1}{\beta R}dR\right]$$

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Results

$$e_T = T^2\left[\left(\frac{1}{\beta^2}\ln\frac{R}{R_0}\right)^2 e_\beta{}^2 + \left(\frac{1}{\beta R}\right)^2 e_R{}^2\right]^{1/2}$$

	Nom. Value	error%	error	error term
β	4261	1%	42.61	0.23
R (Ω)	3,700,000	10%	370000	1.75
T (K)	273.14			**1.77**
T_0 (K)	298.15			
R_0 (Ω)	1000000			

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What if you hate calculus?

$$T = \frac{1}{\frac{1}{T_0} + \frac{1}{\beta}\ln\frac{R}{R_0}}$$

- You can do the whole thing on a spreadsheet with no calculus.

Value	Nominal	β + 1%	β – 1%	R + 10%	R – 10%
T_0 (K)	298.15	298.15	298.15	298.15	298.15
R_0 (Ω)	1,000,000	1,000,000	1,000,000	1,000,000	1,000,000
β	4261	4303.61	4218.39	4261	4261
R (Ω)	3,700,000	3,700,000	3,700,000	4,070,000	3,330,000
T (K)	273.14	273.37	272.91	271.49	275.00
ΔT (K)		0.46		-3.52	
error (K)	**3.55**				
±error (K)	**1.77**				

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Steinhart-Hart Example

$$T = \frac{1}{A_1 + B_1 \ln\frac{R}{R_{ref}} + C_1\left(\ln\frac{R}{R_{ref}}\right)^2 + D_1\left(\ln\frac{R}{R_{ref}}\right)^3}$$

$$dT = \frac{-\left[dA_1 + (\ln R)dB_1 + (\ln R)^2 dC_1 + (\ln R)^3 dD_1 + \left\{\frac{B_1 + 2C_1\left(\ln\frac{R}{R_{ref}}\right) + 3D_1\left(\ln\frac{R}{R_{ref}}\right)^2}{R}\right\}dR\right]}{\left[A_1 + B_1 \ln\frac{R}{R_{ref}} + C_1\left(\ln\frac{R}{R_{ref}}\right)^2 + D_1\left(\ln\frac{R}{R_{ref}}\right)^3\right]^2}$$

$$dT = -T^2\left[dA_1 + (\ln R)dB_1 + (\ln R)^2 dC_1 + (\ln R)^3 dD_1 + \left\{\frac{B_1 + 2C_1\left(\ln\frac{R}{R_{ref}}\right) + 3D_1\left(\ln\frac{R}{R_{ref}}\right)^2}{R}\right\}dR\right]$$

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$$e_T = T^2\left[\left[e_{A_1}\right]^2 + \left[\left(\ln\frac{R}{R_{ref}}\right)e_{B_1}\right]^2 + \left[\left(\ln\frac{R}{R_{ref}}\right)^2 e_{C_1}\right]^2 + \left[\left(\ln\frac{R}{R_{ref}}\right)^3 e_{D_1}\right]^2 + \left[\left\{\frac{B_1 + 2C_1(\ln R) + 3D_1(\ln R)^2}{R}\right\}e_R\right]^2\right]^{1/2}$$

	Nom. Value	error%	error	error term
A_1	3.354016E-03	0.1%	3.35402E-06	0.318
B_1	2.460382E-04	1%	2.46038E-06	-0.104
C_1	3.405377E-06	1%	3.40538E-08	0.001
D_1	1.034240E-07	1%	1.03424E-09	0.000
R (Ω)	64080	3%	1922.4	0.692
T (K)	308.15			**0.769**
R_{ref}	100,000			

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Calculus free style

$$T = \frac{1}{A_1 + B_1 \ln\frac{R}{R_{ref}} + C_1\left(\ln\frac{R}{R_{ref}}\right)^2 + D_1\left(\ln\frac{R}{R_{ref}}\right)^3}$$

Value	Nominal	A_1 +0.1%	A_1 –0.1%	B_1 +1%	B_1 –1%	C_1 +1%	C1 –1%	D_1 +1%	D_1 –1%	R +3%	R –%
A_1	0.003354016	0.00335737	0.003350662	0.003354016	0.003354016	0.003354016	0.003354016	0.003354016	0.003354016	0.003354016	0.003354016
B_1	0.000246038	0.000246038	0.000246038	0.000248499	0.000243578	0.000246038	0.000246038	0.000246038	0.000246038	0.000246038	0.000246038
C_1	3.40538E-06	3.40538E-06	3.40538E-06	3.40538E-06	3.40538E-06	3.43943E-06	3.37132E-06	3.40538E-06	3.40538E-06	3.40538E-06	3.40538E-06
D_1	1.03424E-07	1.03424E-07	1.03424E-07	1.03424E-07	1.03424E-07	1.03424E-07	1.03424E-07	1.04458E-07	1.0239E-07	1.03424E-07	1.03424E-07
R_{ref}	100,000	100,000	100,000	100,000	100,000	100,000	100,000	100,000	100,000	100,000	100,000
R	64,080	64,080	64,080	64,080	64,080	64,080	64,080	64,080	64,080	66,002	62,158
T_{calc}	308.15	307.83	308.47	308.25	308.04	308.15	308.15	308.15	308.15	307.47	308.85
ΔT		-0.64		0.21		0.00		0.00		-1.39	
error	**1.539**										
error ±	**0.769**										

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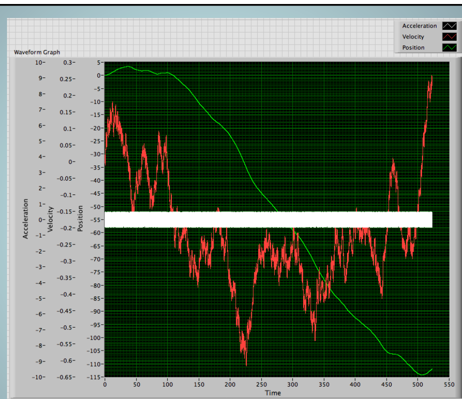
What happens to Integrals?



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Finite Precision?

- Calculate or determine the quantization range, q, e.g., for a ±5V, 12-bit DAQ:

$$q = 10\text{V}\frac{1}{2^{12}} = \frac{10\text{V}}{4096} = 0.027\text{V}$$

- If $S > \frac{10q}{\sqrt{12}}$ ignore quantization
- If $\frac{q}{\sqrt{12}} < S < \frac{10q}{\sqrt{12}}$ include as $S_{used} = \sqrt{S^2 + \frac{q^2}{12}}$
- If $S < \frac{q}{\sqrt{12}}$ confidence interval is $\pm q/2$

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Resources

- Data Analysis Lecture Notes
  - http://www.eng.hmc.edu/NewE80/PDFs/DataAnalysisLecNotes2015.pdf
- NIST Engineering Statistics Handbook
  - http://www.itl.nist.gov/div898/handbook/
- *ISO Guide to the Expression of Uncertainty in Measurement*
  - http://www.iso.org/sites/JCGM/GUM/JCGM100/C045315e-html/C045315e.html?csnumber=50461

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