Auntie Spark's Guide to First-Order Time Constant Measurement

Fall 2014

A first-order system can be characterized by its time constant τ. Simple *RC* or *RL* circuits are examples of first-order systems. Thermometers and thermocouples are often modeled as first-order systems, as are well-stirred mixing tanks. Such systems are characterized experimentally by measuring the time constant, τ. Because all first-order systems exhibit the same response to a given input, a first-order *RC* circuit can be used to demonstrate the behavior of all first-order systems. With a proper choice of *R* and *C*, this circuit can be used to study the behavior of first-order mechanical or thermal systems that are much more difficult to build and characterize in the lab. Therefore, we'll illustrate several methods to measure the time constant, τ, of an *RC* circuit.

RC circuit

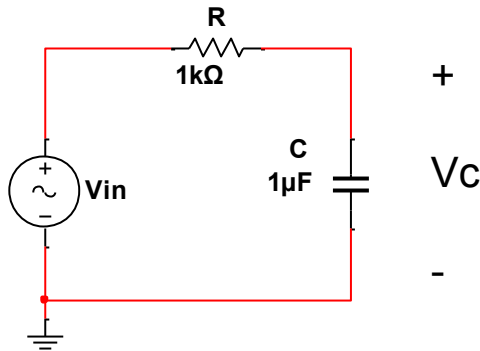Let's use an *RC* series circuit. The voltage across the capacitor *Vc(t)* is the output signal.

Fig.1 RC circuit

Construct the *RC* circuit
- Choose the proper *R* and *C* for the first-order system of interests.
- Connect the circuit on a breadboard
- A function generator will be used to generate the input signal, and the output voltage will be observed with an oscilloscope.
- If *R* and *C* are known or measured prior to the circuit measurement, the expected time constant can be calculated as $\tau = RC$. The measurement can be used to validate the calculated result. In addition, if the component values are not known, measuring the time constant allows for calculation of the product of *R* and *C*.

There are two main methods: time-domain method based on the step response and frequency-domain method based on the frequency response.

Step response of *RC* circuit-time domain method:

One way to extract the time constant τ is by measuring the step response of the *RC* circuit in the time domain. Assume zero initial condition, the step response is given by

$V_c(t) = V_{in}(1 - e^{-\frac{t}{\tau}})$, where V_{in} is the input step voltage. In particular, at $t = \tau$, $V_c(\tau) = V_{in}(1 - e^{-1}) = 0.632V_{in}$, i.e. the output reaches 63.2% of the final steady state value, V_{in}. For example, if the capacitor voltage response is given by Fig. 2 for a step input of V_{in}=5V, we can examine the time at which V_c reaches 63.2%×5 = 3.16V. This time should correspond to $\tau = 0.1$ sec from the result. Since the functional relationship between $V_c(t)$ and t is given, measuring the voltage at any time should also allow you to extract/calculate the time constant τ. In practice, measurements of $V_c(t)$ outside the range of 10% of V_{in} to 90% of V_{in} introduce unacceptable errors into the calculated τ. It is always good practice to take multiple measurements to minimize measurement error. Alternatively least-square curving fitting method can be used to find τ using all of the V-vs-t data (See Auntie Spark's Guide to using LabVIEW to measure first-order time constants for an example). In addition, at the initial time $t = 0$, the initial slope of the V-vs-t data, $\frac{dv_c}{dt}(0) = \frac{V_{in}}{\tau}$. Either the slope can be measured and the time constant, τ, calculated, or the initial-slope line can be extended to the intersection with the steady-state voltage and the time constant determined by the intersection. In the figure below, the slope method leads to the same time constant, $\tau = 0.1$ sec.
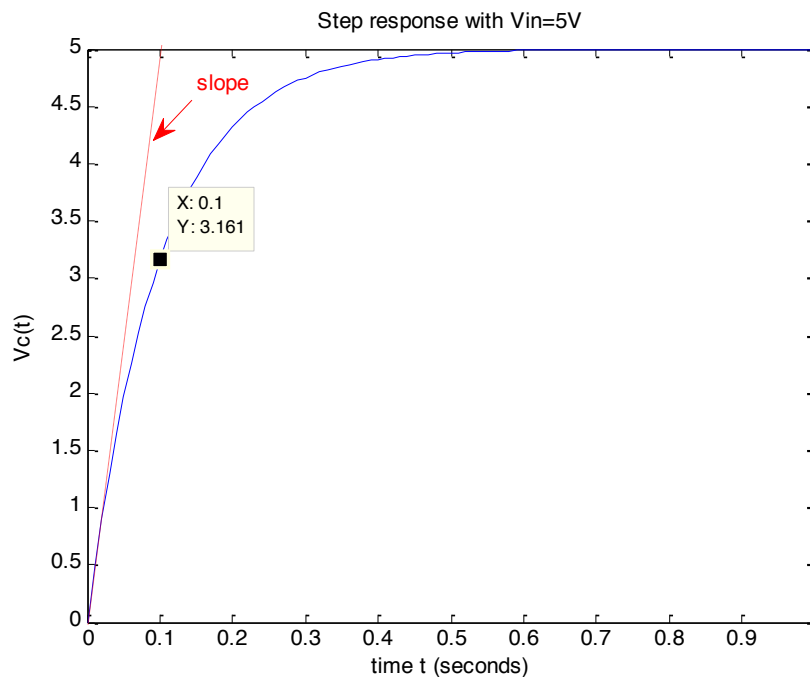

Fig.2 Step response when V_{in}=5V

Experimentally, this measurement requires a function generator for the input voltage and an oscilloscope or DAQ and computer to display the output voltage versus time. A square wave with a period greater than 10 time constants suffices for the step input. Visually, choose a frequency that allows for a complete step response. If the peak-to-peak amplitude of the square wave is given as A, then either the rising edge $V_c(t) = -\frac{A}{2} + A\left(1 - e^{-\frac{t}{\tau}}\right)$, or the falling edge $V_c(t) = \frac{A}{2} + Ae^{-\frac{t}{\tau}}$, can be used to measure the time constant. Fig.3 is an experimental example that shows the typical response to a square

wave with $A = 5$. We choose a period of $T = 1$ second. On the rising edge, the time at which $\frac{Y-(-2.5)}{2.5-(-2.5)} = 63.2\% \rightarrow Y = 0.66$ corresponds to the time constant $\tau = 0.1$ sec. On the falling edge, similarly, the time at which $\frac{Y-(2.5)}{2.5-(-2.5)} = e^{-1} = 36.8\% \rightarrow Y = -0.66$ corresponds to time constant $\tau = 1.1 - 1 = 0.1$ sec.
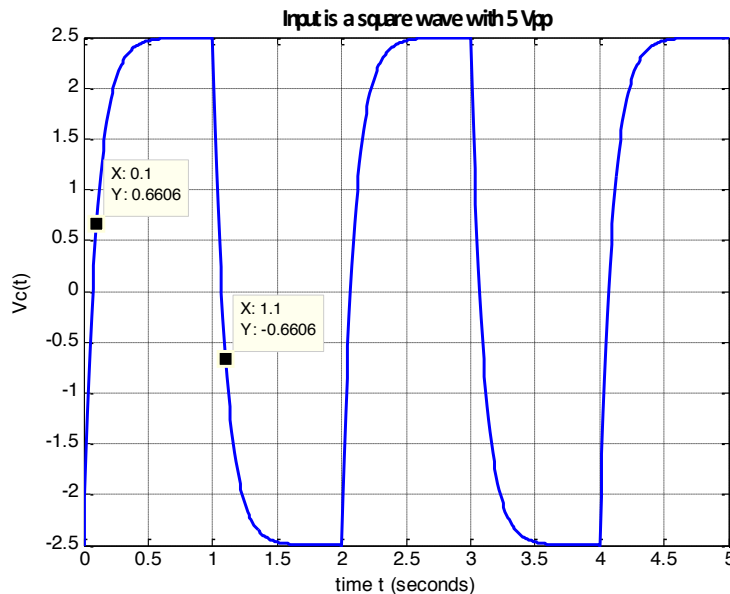

Fig.3 RC circuit response to a square wave voltage input, A=5V

Frequency response of RC circuit-frequency domain method:

An alternative method of extracting the time constant, τ, is measuring the frequency response function (FRF) of the circuit. A Bode plot is a convenient way to plot magnitude and phase of the FRF as a function of frequency ω (rad/s) or f (Hz). A typical first-order RC circuit's Bode plot is given in Figure 4.

Bode Diagram
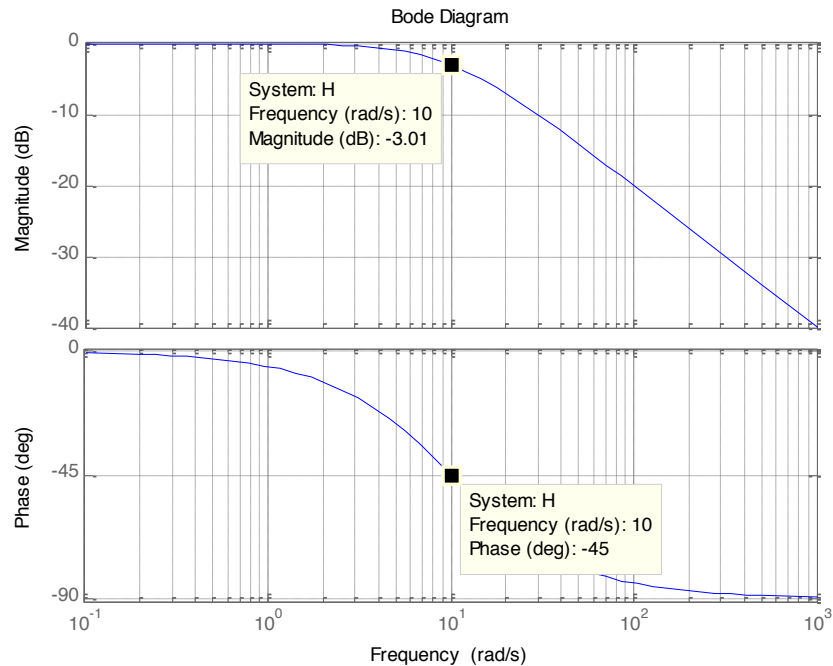
In this approach a similar experimental setup is required; however, the frequency of the input or function generator is adjusted to obtain the 3dB corner or cutoff frequency of the system. Because the 3dB corner frequency is related to τ by $\omega_{3dB} = \frac{1}{\tau}$ or $f_{3dB} = \frac{1}{2\pi\tau}$. In the RC circuit shown, the 3dB frequency corresponds to when the capacitor voltage drops down to $\frac{1}{\sqrt{2}} = 0.707$ of the low frequency values. In addition, phase measurements can be used to identify the corner frequency too. If comparing input voltage and output voltage, the frequency at which the phase difference reaches exactly $-\frac{\pi}{4}$ corresponds to 3dB corner frequency.

- Note that a function generator displays frequency in Hz instead of rad/s.
- If a full bode plot is to be constructed using such measurements, make sure to plot $Lm = 20log_{10} \left| \frac{V_{out}}{V_{in}} \right| (dB)$ for the magnitude in dB, and ω (or f) on a logarithm scale.
- The cursor option on the scope can be very handy when measuring when the amplitude ratio has dropped (or increased) to 0.707 or for measuring the phase difference between two signals.

The frequency-response-function method is easy to implement for electrical systems in the lab, but is more challenging for mechanical or thermal systems because the excitation frequency is much more difficult to control.