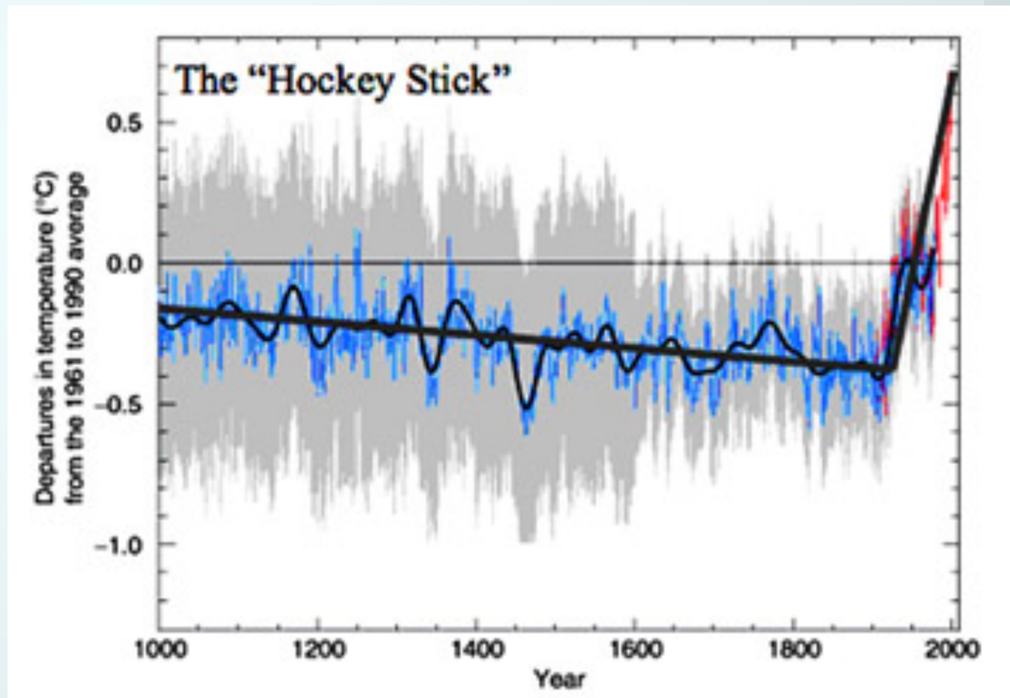
Measuring Things

E80 Spring 2011

Why do we care?

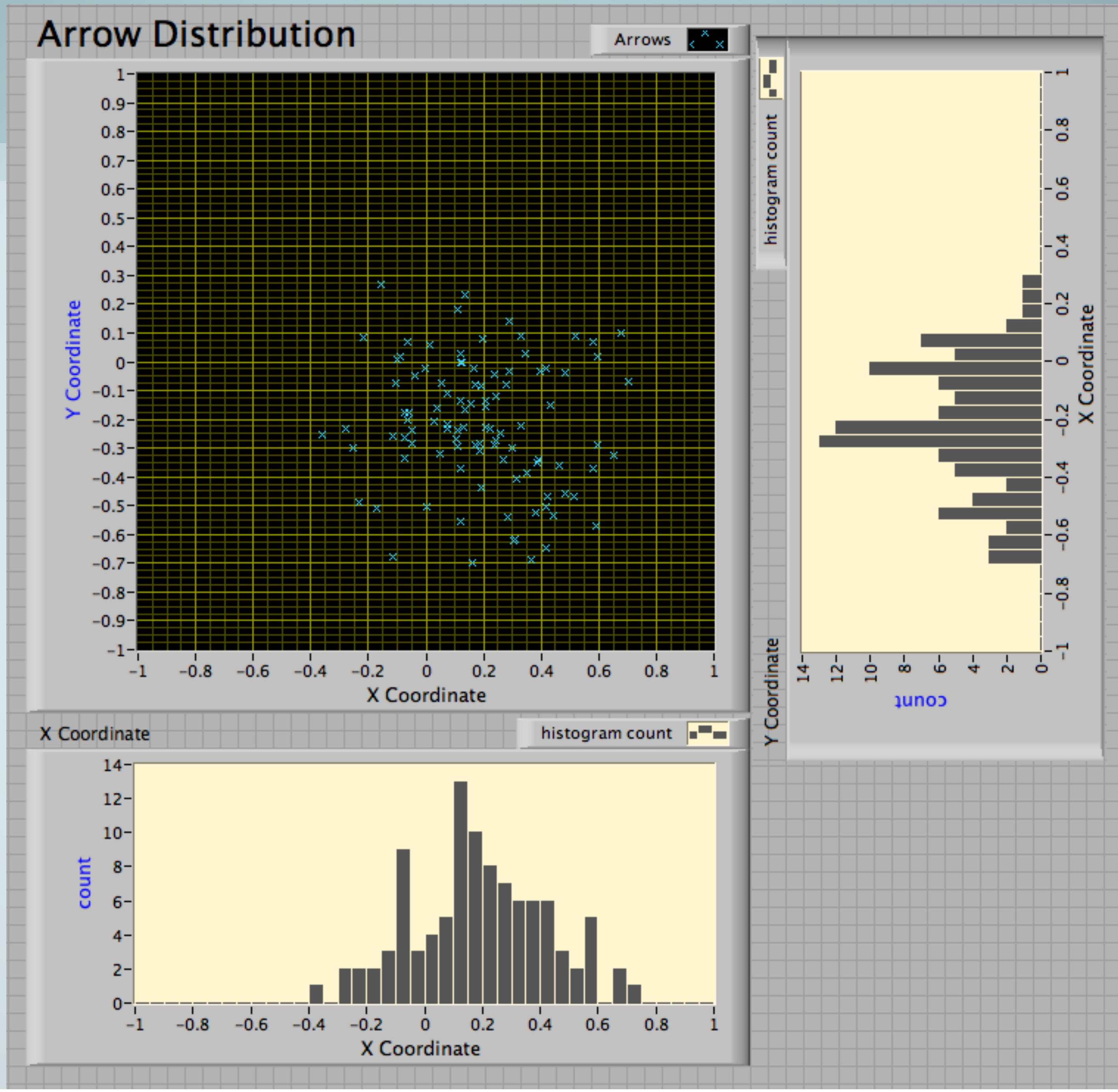
- Who is this man?



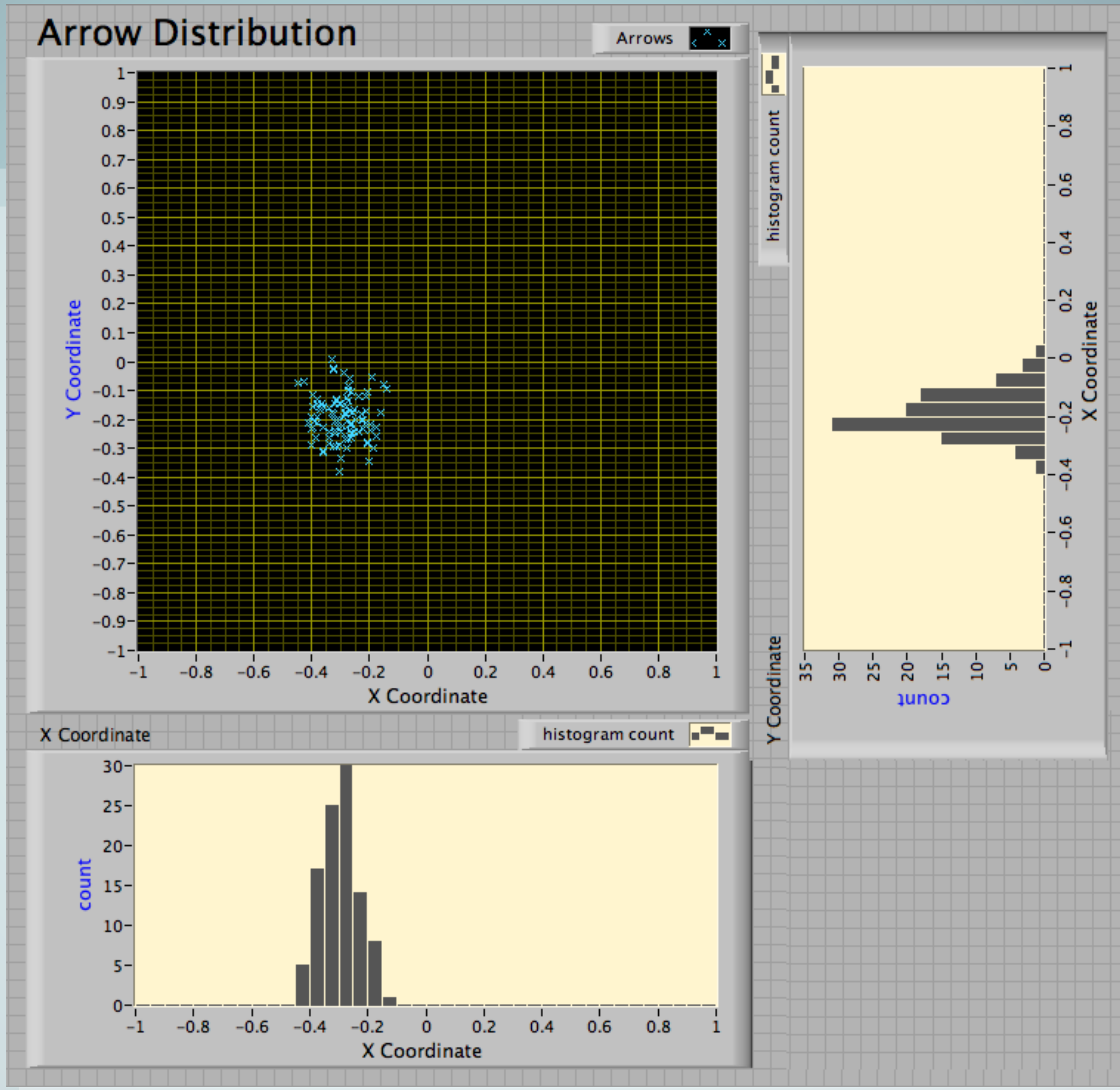
Your Mission

- Mythbuster Tech Advisor
- Testing Arrows for a crossbow

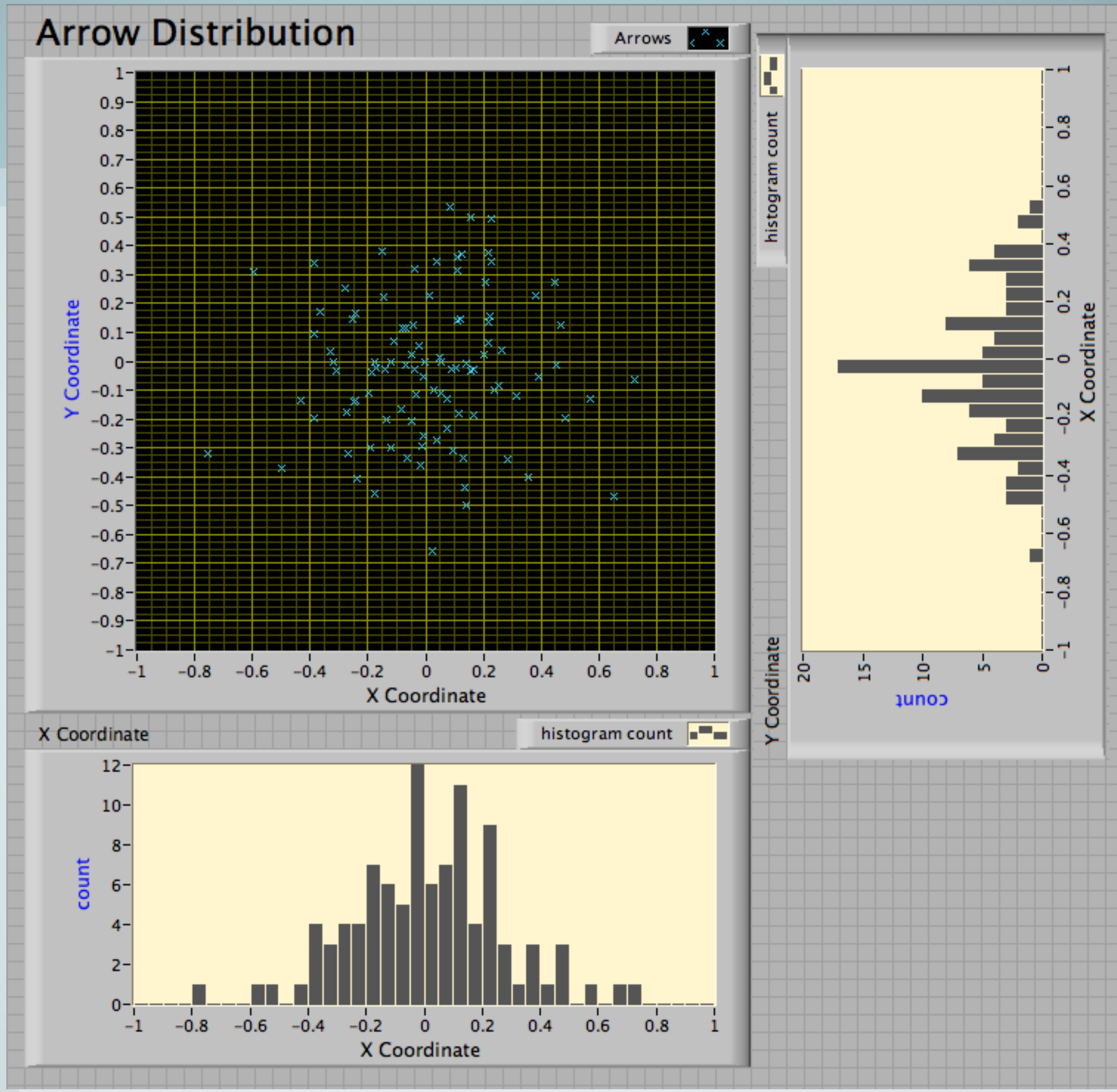
Brand 1 Arrows



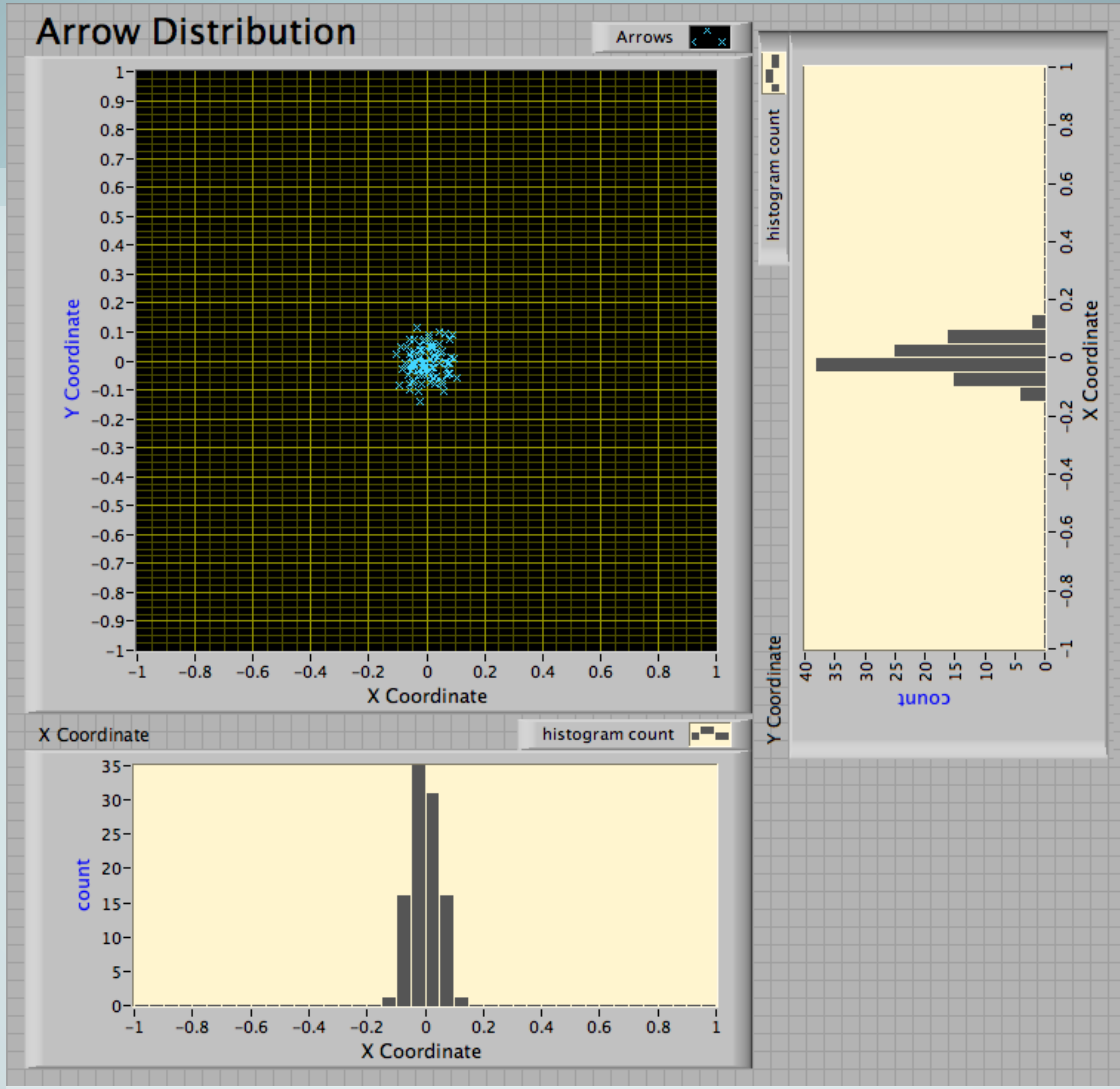
Brand 2 Arrows



Brand 3 Arrows



Brand 4 Arrows



Crossbow Questions

- Where is the center of the distribution, i.e., what is the aiming point?
- How certain are you?
- If we aim the crossbow at the apple, how likely is Buster™ to get an arrow in the head?

Measured Value

- Where is the center of the distribution, i.e., where should we put the target?
- Use the mean.

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

Error Estimate

- How certain are you? $\pm\lambda$
- Use Standard Error and Student's *t*-test

$$\lambda = tSE = \frac{tS}{\sqrt{N}}$$

$$S = \sqrt{S^2}$$

$$S^2 \equiv \frac{1}{N-1}\sum_{i=1}^{N} e_i^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \bar{x}\right)^2$$

Students *t*-test

SIGNIFICANCE LEVEL FOR TWO-TAILED TEST						
df	.20	.10	.05	.02	.01	.001
1	3.078	6.314	12.706	31.821	63.657	636.619
2	1.886	2.920	4.303	6.965	9.925	31.598
3	1.638	2.353	3.182	4.541	5.841	12.941
4	1.533	2.132	2.776	3.747	4.604	8.610
5	1.476	2.015	2.571	3.365	4.032	6.859
10	1.372	1.812	2.228	2.764	3.169	4.587
20	1.325	1.725	2.086	2.528	2.845	3.850
30	1.310	1.697	2.042	2.457	2.750	3.646
40	1.303	1.684	2.021	2.423	2.704	3.551
60	1.296	1.671	2.000	2.390	2.660	3.460
120	1.289	1.658	1.980	2.358	2.617	3.373
∞	1.282	1.645	1.960	2.326	2.576	3.291

Spread of Data

- If we aim the crossbow at the apple, how likely is Buster™ to get an arrow in the head?
- Use Standard Deviation

$$S = \sqrt{S^2}$$

$$S^2 \equiv \frac{1}{N-1}\sum_{i=1}^{N} e_i^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \overline{x}\right)^2$$

Linear Fits

- How about fitting lines to sets of data?

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

- Use Linear Least Squares fit.

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{N}(x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^{N}(x_i - \overline{x})^2}$$

$$\hat{\beta}_0 = \overline{y} - \hat{\beta}_1 \overline{x}$$

How Certain Are We?

- How good are the Betas?

$$S_e = \sqrt{\frac{SSE}{N-2}} = \sqrt{\frac{\sum_{i=1}^{N} e_i^2}{N-2}} \qquad df = N-2$$

$$S_{\beta_0} = S_e \sqrt{\frac{1}{N} + \frac{\overline{x}^2}{\sum_{i=1}^{N}(x_i - \overline{x})^2}} \qquad \lambda_{\beta_0} = tS_{\beta_0}$$

$$S_{\beta_1} = S_e \sqrt{\frac{1}{\sum_{i=1}^{N}(x_i - \overline{x})^2}} \qquad \lambda_{\beta_1} = tS_{\beta_1}$$

Why? Because!

- How certain is the calculated value of y for a chosen x?

$$S_y = S_e \sqrt{\frac{1}{N} + \frac{(x-\bar{x})^2}{\sum_{i=1}^{N}(x_i-\bar{x})^2}} \qquad \lambda_y = tS_y$$

- If we experimentally set x, how will the experimental y's spread (The arrow in the head question)?
- Use S_e like you previously used S.

What About Functions?

- What if you want *T*?

$$T = \frac{1}{a + b\ln R + c(\ln R)^3}$$

- and you measure *a*, *b*, *c*, and *R*?

Error Propagation

- For $F = F(x, y, z, \cdots)$

$$F - F_{true} = \frac{\partial F}{\partial x}\left(x - x_{true}\right) + \frac{\partial F}{\partial y}\left(y - y_{true}\right) + \frac{\partial F}{\partial z}\left(z - z_{true}\right) + \cdots$$

- Let $\varepsilon_x = x - x_{true}$

$$\varepsilon_F = \frac{\partial F}{\partial x}\varepsilon_x + \frac{\partial F}{\partial y}\varepsilon_y + \frac{\partial F}{\partial z}\varepsilon_z + \cdots$$

$$\varepsilon_F = \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 \varepsilon_x^2 + \left(\frac{\partial F}{\partial y}\right)^2 \varepsilon_y^2 + \left(\frac{\partial F}{\partial z}\right)^2 \varepsilon_z^2 + \cdots}$$

Steinhart-Hart Example

$$T = \frac{1}{a + b\ln R + c\left(\ln R\right)^3}$$

$$dT = \frac{-\left[da + \left(\ln R\right)db + \left(\ln R\right)^3 dc + \left\{\frac{b + 3c\left(\ln R\right)^3}{R}\right\}dR\right]}{\left[a + b\ln R + c\left(\ln R\right)^3\right]^2}$$

$$dT = -T^2\left[da + \left(\ln R\right)db + \left(\ln R\right)^3 dc + \left\{\frac{b + 3c\left(\ln R\right)^3}{R}\right\}dR\right]$$

$$e_T = T^2\left[\left[e_a\right]^2 + \left[(\ln R)e_b\right]^2 + \left[(\ln R)^3 e_c\right]^2 + \left[\left\{\frac{b+3c(\ln R)^3}{R}\right\}e_R\right]^2\right]^{1/2}$$

	Value	error %	error	error term (K)
a	1.126E-03	1%	1.126E-05	1.00
b	2.346E-04	1%	2.346E-06	1.92
c	8.619E-08	1%	8.619E-10	0.06
R(Ω)	10000	10%	1000	3.88
T(K)	**298.15**	**1.5%**		**4.45**

Steinhart-Hart Example

$$T = \frac{1}{\frac{1}{T_0} + \frac{1}{\beta}\ln\frac{R}{R_0}}$$

$$dT = \frac{\left[\left(\frac{1}{\beta^2}\ln\frac{R}{R_0}\right)d\beta - \frac{1}{\beta R}dR\right]}{\left[\frac{1}{T_0} + \frac{1}{\beta}\ln\frac{R}{R_0}\right]^2}$$

$$dT = T^2\left[\left(\frac{1}{\beta^2}\ln\frac{R}{R_0}\right)d\beta - \frac{1}{\beta R}dR\right]$$

Results

$$e_T = T^2\left[\left(\frac{1}{\beta^2}\ln\frac{R}{R_0}\right)^2 e_\beta{}^2 - \left(\frac{1}{\beta R}\right)^2 e_R{}^2\right]^{1/2}$$

	Value	**error %**	**error**	**error term (K)**
T0	298.15	0%		0
R0	1,000,000	0%		0
Beta	4261	1%	42.61	0.10
R (Ω)	3,700,000	10%	370,000	-1.75
T (K)	**273.14**	**0.6%**		**1.75**

Finite Precision?

- Calculate or determine the quantization range, *q*, e.g., for a ±5V, 12-bit DAQ:

$$q = 10\text{V}\frac{1}{2^{12}} = \frac{10\text{V}}{4096} = 0.027\text{V}$$

- If $S > \frac{10q}{\sqrt{12}}$ ignore quantization

- If $\frac{q}{\sqrt{12}} < S < \frac{10q}{\sqrt{12}}$ include as $S_{used} = \sqrt{S^2 + \frac{q^2}{12}}$

- If $S < \frac{q}{\sqrt{12}}$ confidence interval is $\pm q/2$