Addition of Drag and Gravity to the 1-D Rocket Equation

We'll take as given the derivation at the end of the Rocket Equation video

$$m\frac{dv}{dt}\left(=ma\right)=\frac{dm}{dt}v_e, \tag{1.1}$$

where v_e, the exit velocity of the propellant, is a vector quantity in the direction opposite the rocket. We'll assume that the rocket stays vertical and flies under thrust in the positive *x* (or *z*) direction, so vector quantities are specified as the magnitude and a positive or negative sign to indicate up or down. We'll first do a quick derivation of the relationship between specific impulse and exit velocity. Starting from the definition that

$$T=v_e\frac{dm}{dt}, \tag{1.2}$$

and for a given motor v_e is constant (not quite true but true enough), we have that

$$\begin{aligned} dm &= -\frac{1}{|v_e|}Tdt \\ \int_{m_0}^{m} dm &= -\frac{1}{|v_e|}\int_{t_0}^{t} Tdt \\ \Delta m &= -\frac{1}{|v_e|}I=-m_p \end{aligned} \tag{1.3}$$

resulting in the impulse *I* being given by

$$I=m_p\left|v_e\right|, \tag{1.4}$$

where m_p is the mass of the propellant. Rearranging, we have the exit velocity is

$$\left|v_e\right|=\frac{I}{m_p}. \tag{1.5}$$

There are two definitions for specific impulse, I_{sp}. The first is the impulse divided by the propellant mass, I/m_p, with SI units of

$$\frac{\mathrm{Ns}}{\mathrm{kg}}=\frac{\mathrm{m}}{\mathrm{s}}, \tag{1.6}$$

and the second is impulse divided by the propellant weight in standard gravity, $I/\left(m_p g_0\right)$, with SI units of

$$\frac{\mathrm{Ns}}{\mathrm{kg}\frac{\mathrm{m}}{\mathrm{s}^2}} = \mathrm{s} \tag{1.7}$$

You can tell whether you need to multiply by g_0 or not to get v_e by checking the units of I_{sp}. Substituting in thrust for $\dot{m}v_e$ in Equation (1.1), we have

$$m\frac{dv}{dt}(= ma) = T \tag{1.8}$$

At this point, we can simply add the gravity force, $-mg$ (with m now a function of time), and the drag force, $F_D = -\frac{1}{2}C_D A_p \rho v|v|$, to the rocket equation.

$$a = \dot{v} = \frac{T}{m} - g - \frac{1}{2}C_D A_p \frac{\rho}{m} v|v| \tag{1.9}$$

The result is the same as Equation 1.3 of the analytical model, but now with m as a function of time. There are several analytical solutions to the rocket equation, depending on what you assume about the thrust curve, the importance of drag, and whether to include gravity or not. In general, when you include a non-constant thrust curve, drag, and properties that change with altitude, you need to numerically integrate two coupled differential equations. We include integration for altitude or position as a third equation. The three are

$$\dot{v} = \frac{T}{m} - g - \frac{1}{2}C_D A_p \frac{\rho}{m} v|v| \tag{1.10}$$

$$\dot{m} = \frac{T}{v_e} \tag{1.11}$$

and

$$\dot{x} = v \tag{1.12}$$

This system of equations can be integrated with any handy D.E. solver, such as the Runge-Kutta 4th-order method. The method requires some way to read in the thrust curve, and parameter values or functions of velocity or altitude for C_D, ρ, and g.