1-D Analytical Model

Assume that the thrust is constant from $t = 0$ to $t = t_1$ at a value of $T_1$. The initial velocity is $v_0 = 0$. Then from $t = t_1$ to $t = t_2$ the thrust is constant at a value of $T_2$. Finally, from $t = t_2$ to $t = t_d$ (when the parachute deploys), the thrust is constant at $T_0 = 0$. For the drag force we assume a constant drag coefficient, $C_D$. The drag force is then

$$F_D = \frac{1}{2} C_D A_P \rho v^2 \ . \tag{1.1}$$

The force balance on the rocket is

$$ma = T_x - \frac{1}{2} C_D A_p \rho v |v| - mg \tag{1.2}$$

Where $x$ is 1, 2, or 0 and the absolute value sign takes care of whether the rocket is rising or falling. Division through by the mass gives

$$a = \frac{T_x}{m} - g - \frac{1}{2} C_D A_p \frac{\rho}{m} v |v| \tag{1.3}$$

To make computations easier, we'll calculate the velocity when the acceleration is 0. With no thrust, the velocity is the terminal velocity, $v_t$. For the thrust case, we'll call it the maximum velocity, $v_{\max}$. The terminal velocity is

$$v_t = \sqrt{\frac{2mg}{\rho C_D A_p}} \tag{1.4}$$

And the maximum velocity is

$$v_{\max} = \sqrt{\frac{2(T_x - mg)}{\rho C_D A_p}} \tag{1.5}$$

The acceleration can then be written as

$$a = \left(\frac{T_x - mg}{m}\right)\left(1 - \frac{v^2}{v_{\max}^2}\right) = b\left(1 - \frac{v^2}{v_{\max}^2}\right). \tag{1.6}$$

Written in differential equation form it becomes

$$\frac{dv}{dt} = b\left(1 - \frac{v^2}{v_{\max}^2}\right) \tag{1.7}$$

Which is separable to

$$\frac{dv}{\left(1-\frac{v^2}{v_{max}^2}\right)} = bdt. \tag{1.8}$$

Integrating

$$\int_{v_0}^{v_f} \frac{dv}{\left(1-\frac{v^2}{v_{max}^2}\right)} = b\int_{t_0}^{t_f} dt \tag{1.9}$$

$$v_{max}\tanh^{-1}\frac{v}{v_{max}}\Bigg|_{v_0}^{v_f} = b\left(t_f - t_0\right) \tag{1.10}$$

Or

$$v_f = v_{max}\tanh\left[\frac{b}{v_{max}}\left(t_f - t_0\right) + \tanh^{-1}\frac{v_0}{v_{max}}\right] \tag{1.11}$$

However, if $v_0$ is greater than $v_{max}$, then the integral becomes

$$v_{max}\coth^{-1}\frac{v}{v_{max}}\Bigg|_{v_0}^{v_f} = b\left(t_f - t_0\right) \tag{1.12}$$

Or

$$v_f = v_{max}\coth\left[\frac{b}{v_{max}}\left(t_f - t_0\right) + \coth^{-1}\frac{v_0}{v_{max}}\right] \tag{1.13}$$

The physical difference is that if $v_0$ is less than $v_{max}$, the rocket will accelerate, but if $v_0$ is greater than $v_{max}$, the rocket will decelerate.

$v_0$ and $t_0$ are initial conditions, but $v_f$ is the velocity at $t_f$, which is an arbitrary time within the specified interval (we could have used dummy variables and integrated to *t*). We can therefore write for $v_0 < v_{max}$,

$$\frac{dx}{dt} = v_{max}\tanh\left[\frac{b}{v_{max}}\left(t - t_0\right) + \tanh^{-1}\frac{v_0}{v_{max}}\right] \tag{1.14}$$

Which is again separable into

$$dx = v_{\max} \tanh\left[ \frac{b}{v_{\max}}(t - t_0) + \tanh^{-1} \frac{v_0}{v_{\max}} \right] dt \tag{1.15}$$

Integrating from $x_0$ to $x$ and from $t_0$ to $t$ yields:

$$x - x_0 = \frac{v_{\max}^2}{b} \left\{ \ln\left[ \cosh\left( \frac{b}{v_{\max}}(t - t_0) + \tanh^{-1} \frac{v_0}{v_{\max}} \right) \right] - \ln\left[ \cosh\left( \tanh^{-1} \frac{v_0}{v_{\max}} \right) \right] \right\} \tag{1.16}$$

The equivalent process for $v_0 > v_{\max}$ is

$$\frac{dx}{dt} = v_{\max} \coth\left[ \frac{b}{v_{\max}}(t_f - t_0) + \coth^{-1} \frac{v_0}{v_{\max}} \right] \tag{1.17}$$

Which is again separable into

$$dx = v_{\max} \coth\left[ \frac{b}{v_{\max}}(t_f - t_0) + \coth^{-1} \frac{v_0}{v_{\max}} \right] dt \tag{1.18}$$

Integrating from $x_0$ to $x$ and from $t_0$ to $t$ yields:

$$x - x_0 = \frac{v_{\max}^2}{b} \left\{ \ln\left[ \sinh\left( \frac{b}{v_{\max}}(t - t_0) + \coth^{-1} \frac{v_0}{v_{\max}} \right) \right] - \ln\left[ \sinh\left( \coth^{-1} \frac{v_0}{v_{\max}} \right) \right] \right\} \tag{1.19}$$

During the coasting decent from apogee, Equations (1.11) and (1.16) apply with $v_{\max}$ being replaced with $v_t$ and $b$ being replaced by $-g$.

The time from apogee to impact/landing is given by solving equation 1.16 from apogee to the ground for the time. With the relevant substitutions, we are solving

$$\begin{aligned} 0 - x_0 &= \frac{v_t^2}{-g} \left\{ \ln\left[ \cosh\left( \frac{g}{v_t}(t - t_0) + \tanh^{-1} \frac{0}{v_t} \right) \right] - \ln\left[ \cosh\left( \tanh^{-1} \frac{0}{v_t} \right) \right] \right\} \\ &= \frac{v_t^2}{-g} \left\{ \ln\left[ \cosh\left( \frac{g}{v_t}(t - t_0) \right) \right] \right\} \end{aligned} \tag{1.20}$$

for $t - t_0$, the time from apogee to impact/landing. We used $g$ instead of $-g$ inside the cosh function because it is an even function, and we want the time to be positive. Solving, we get

$$t - t_0 = \frac{v_t}{g} \cosh^{-1}\left[ \exp\left( \frac{g x_0}{v_t^2} \right) \right]. \tag{1.21}$$

For large values of $\frac{gx_0}{v_t^2}$, the exponential may overflow, and not permit calculation of the time. If so, you may need to look up the inverse hyperbolic cosine in terms of natural log and use an approximate expression. In that case, for $\frac{gx_0}{v_t^2} > 100$, we get

$$t - t_0 = \frac{x_0}{v_t} + \frac{v_t}{g}\ln 2. \tag{1.22}$$

And then using the time in Equation 1.11 to solve for the impact/landing velocity, we get

$$\begin{aligned} v_f &= v_t \tanh\left[\frac{g}{v_t}\left(t_f - t_0\right) + \tanh^{-1}\frac{0}{v_t}\right] = v_t \tanh\left\{\frac{g}{v_t}\frac{v_t}{g}\cosh^{-1}\left[\exp\left(\frac{gx_0}{v_t^2}\right)\right]\right\} \\ &= v_t \tanh\left\{\cosh^{-1}\left[\exp\left(\frac{gx_0}{v_t^2}\right)\right]\right\} \end{aligned} \tag{1.23}$$

which asymptotically approaches $v_t$.

Going back to our derivation, during the coasting ascent to apogee, Equation (1.6) becomes

$$a = -g\left(1 + \frac{v^2}{v_t^2}\right) \tag{1.24}$$

Which separates and integrates to

$$\tan^{-1}\frac{v}{v_t} - \tan^{-1}\frac{v_0}{v_t} = -\frac{g}{v_t}\left(t - t_0\right) \tag{1.25}$$

or

$$v = v_t \tan\left[\frac{-g}{v_t}\left(t - t_0\right) + \tan^{-1}\frac{v_0}{v_t}\right]. \tag{1.26}$$

To get the position, separate and integrate Equation (1.26).

$$x - x_0 = \frac{v_t^2}{g}\left(\ln\left\{\cos\left[\frac{g}{v_t}\left(t - t_0\right) - \tan^{-1}\frac{v_0}{v_t}\right]\right\} + \ln\sqrt{1 + \frac{v_0^2}{v_t^2}}\right) \tag{1.27}$$

Apogee, and the time to apogee is found by setting the velocity to 0 in Equation (1.26), solving for the time to apogee and substituting into Equation (1.27).

$$0 = v_t \tan\left[\frac{-g}{v_t}\left(t_{\text{apogee}} - t_0\right) + \tan^{-1}\frac{v_0}{v_t}\right], \tag{1.28}$$

$$\frac{g}{v_t}\left(t_{\text{apogee}} - t_0\right) = \tan^{-1}\frac{v_0}{v_t} \ , \tag{1.29}$$

$$t_{\text{apogee}} - t_0 = \frac{v_t}{g}\tan^{-1}\frac{v_0}{v_t} \ , \tag{1.30}$$

$$\begin{aligned} x_{\text{apogee}} - x_0 &= \frac{v_t^2}{g}\left(\ln\left\{\cos\left[\frac{g}{v_t}\frac{v_t}{g}\tan^{-1}\frac{v_0}{v_t} - \tan^{-1}\frac{v_0}{v_t}\right]\right\} + \ln\sqrt{1+\frac{v_0^2}{v_t^2}}\right) \\ &= \frac{v_t^2}{g}\left\{\ln\left[\cos\left(0\right)\right] + \ln\sqrt{1+\frac{v_0^2}{v_t^2}}\right\} = \frac{v_t^2}{g}\ln\sqrt{1+\frac{v_0^2}{v_t^2}} \end{aligned} . \tag{1.31}$$

When implementing these equations in code, or when solving by hand, remember that they are piecewise function and that the final conditions for one segment, $t_f$, $v_f$, and $x_f$, are the initial conditions for the next segment, $t_0$, $v_0$, and $x_0$