1-D Analytical Model

Assume that the thrust is constant from $t=0$ to $t=t_1$ at a value of T_1. The initial velocity is $v_0=0$. Then from $t=t_1$ to $t=t_2$ the thrust is constant at a value of T_2. Finally, from $t=t_2$ to $t=t_d$ (when the parachute deploys), the thrust is constant at $T_0=0$. For the drag force we assume a constant drag coefficient, C_D. The drag force is then

$$F_D=\frac{1}{2}C_D A_P \rho v^2 \quad . \tag{1.1}$$

The force balance on the rocket is

$$ma=T_x-\frac{1}{2}C_D A_p \rho v|v|-mg \tag{1.2}$$

Where x is 1, 2, or 0 and the absolute value sign takes care of whether the rocket is rising or falling. Division through by the mass gives

$$a=\frac{T_x}{m}-g-\frac{1}{2}C_D A_p \frac{\rho}{m}v|v| \tag{1.3}$$

To make computations easier, we'll calculate the velocity when the acceleration is 0. With no thrust, the velocity is the terminal velocity, v_t. For the thrust case, we'll call it the maximum velocity, $v_{\max}$. The terminal velocity is

$$v_t=\sqrt{\frac{2mg}{\rho C_D A_p}} \tag{1.4}$$

And the maximum velocity is

$$v_{\max}=\sqrt{\frac{2(T_x-mg)}{\rho C_D A_p}} \tag{1.5}$$

The acceleration can then be written as

$$a=\left(\frac{T_x-mg}{m}\right)\left(1-\frac{v^2}{v_{\max}^2}\right)=b\left(1-\frac{v^2}{v_{\max}^2}\right). \tag{1.6}$$

Written in differential equation form it becomes

$$\frac{dv}{dt}=b\left(1-\frac{v^2}{v_{\max}^2}\right) \tag{1.7}$$

Which is separable to

$$\frac{dv}{\left(1-\frac{v^2}{v_{\max}^2}\right)} = b\,dt\,. \tag{1.8}$$

Integrating

$$\int_{v_0}^{v_f} \frac{dv}{\left(1-\frac{v^2}{v_{\max}^2}\right)} = b\int_{t_0}^{t_f} dt \tag{1.9}$$

$$v_{\max} \tanh^{-1} \frac{v}{v_{\max}} \Bigg|_{v_0}^{v_f} = b\left(t_f - t_0\right) \tag{1.10}$$

Or

$$v_f = v_{\max} \tanh\left[\frac{b}{v_{\max}}\left(t_f - t_0\right) + \tanh^{-1}\frac{v_0}{v_{\max}}\right] \tag{1.11}$$

However, if v_0 is greater than $v_{\max}$, then the integral becomes

$$v_{\max} \coth^{-1} \frac{v}{v_{\max}} \Bigg|_{v_0}^{v_f} = b\left(t_f - t_0\right) \tag{1.12}$$

Or

$$v_f = v_{\max} \coth\left[\frac{b}{v_{\max}}\left(t_f - t_0\right) + \coth^{-1}\frac{v_0}{v_{\max}}\right] \tag{1.13}$$

The physical difference is that if v_0 is less than $v_{\max}$, the rocket will accelerate, but if v_0 is greater than $v_{\max}$, the rocket will decelerate.

v_0 and t_0 are initial conditions, but v_f is the velocity at t_f, which is an arbitrary time within the specified interval (we could have used dummy variables and integrated to *t*). We can therefore write for $v_0 < v_{\max}$,

$$\frac{dx}{dt} = v_{\max} \tanh\left[\frac{b}{v_{\max}}\left(t - t_0\right) + \tanh^{-1}\frac{v_0}{v_{\max}}\right] \tag{1.14}$$

Which is again separable into

$$dx = v_{\max} \tanh\left[\frac{b}{v_{\max}}(t - t_0) + \tanh^{-1}\frac{v_0}{v_{\max}}\right]dt \tag{1.15}$$

Integrating from x_0 to x and from t_0 to t yields:

$$x - x_0 = \frac{v_{\max}^2}{b}\left\{\ln\left[\cosh\left(\frac{b}{v_{\max}}(t - t_0) + \tanh^{-1}\frac{v_0}{v_{\max}}\right)\right] - \ln\left[\cosh\left(\tanh^{-1}\frac{v_0}{v_{\max}}\right)\right]\right\} \tag{1.16}$$

The equivalent process for $v_0 > v_{\max}$ is

$$\frac{dx}{dt} = v_{\max} \coth\left[\frac{b}{v_{\max}}(t_f - t_0) + \coth^{-1}\frac{v_0}{v_{\max}}\right] \tag{1.17}$$

Which is again separable into

$$dx = v_{\max} \coth\left[\frac{b}{v_{\max}}(t_f - t_0) + \coth^{-1}\frac{v_0}{v_{\max}}\right]dt \tag{1.18}$$

Integrating from x_0 to x and from t_0 to t yields:

$$x - x_0 = \frac{v_{\max}^2}{b}\left\{\ln\left[\sinh\left(\frac{b}{v_{\max}}(t - t_0) + \coth^{-1}\frac{v_0}{v_{\max}}\right)\right] - \ln\left[\sinh\left(\coth^{-1}\frac{v_0}{v_{\max}}\right)\right]\right\} \tag{1.19}$$

During the coasting decent from apogee, Equations (1.11) and (1.16) apply with $v_{\max}$ being replaced with v_t and b being replaced by $-g$.

The time from apogee to impact/landing is given by solving equation 1.16 from apogee to the ground for the time, and then using the time in Equation 1.11 to solve for the impact/landing velocity. For very long landing times, you may need to look up the inverse hyperbolic cosine in terms of natural log and use an approximate expression.

However, during the coasting ascent to apogee, Equation (1.6) becomes

$$a = -g\left(1 + \frac{v^2}{v_t^2}\right) \tag{1.20}$$

Which separates and integrates to

$$\tan^{-1}\frac{v}{v_t} - \tan^{-1}\frac{v_0}{v_t} = -\frac{g}{v_t}(t - t_0) \tag{1.21}$$

or

$$v = v_t \tan\left[\frac{-g}{v_t}\left(t - t_0\right) + \tan^{-1}\frac{v_0}{v_t}\right] . \tag{1.22}$$

To get the position, separate and integrate Equation (1.22).

$$x - x_0 = \frac{v_t^2}{g}\left(\ln\left\{\cos\left[\frac{g}{v_t}\left(t - t_0\right) - \tan^{-1}\frac{v_0}{v_t}\right]\right\} + \ln\sqrt{1 + \frac{v_0^2}{v_t^2}}\right) \tag{1.23}$$

Apogee, and the time to apogee is found by setting the velocity to 0 in Equation (1.22), solving for the time to apogee and substituting into Equation (1.23).

$$0 = v_t \tan\left[\frac{-g}{v_t}\left(t_{\text{apogee}} - t_0\right) + \tan^{-1}\frac{v_0}{v_t}\right] , \tag{1.24}$$

$$\frac{g}{v_t}\left(t_{\text{apogee}} - t_0\right) = \tan^{-1}\frac{v_0}{v_t} , \tag{1.25}$$

$$t_{\text{apogee}} - t_0 = \frac{v_t}{g}\tan^{-1}\frac{v_0}{v_t} , \tag{1.26}$$

$$\begin{aligned} x_{\text{apogee}} - x_0 &= \frac{v_t^2}{g}\left(\ln\left\{\cos\left[\frac{g}{v_t}\frac{v_t}{g}\tan^{-1}\frac{v_0}{v_t} - \tan^{-1}\frac{v_0}{v_t}\right]\right\} + \ln\sqrt{1 + \frac{v_0^2}{v_t^2}}\right) \\ &= \frac{v_t^2}{g}\left\{\ln\left[\cos\left(0\right)\right] + \ln\sqrt{1 + \frac{v_0^2}{v_t^2}}\right\} = \frac{v_t^2}{g}\ln\sqrt{1 + \frac{v_0^2}{v_t^2}} \end{aligned} . \tag{1.27}$$

When implementing these equations in code, or when solving by hand, remember that they are piecewise function and that the final conditions for one segment, t_f, v_f, and x_f, are the initial conditions for the next segment, t_0, v_0, and x_0

Thrustcurve.org E12

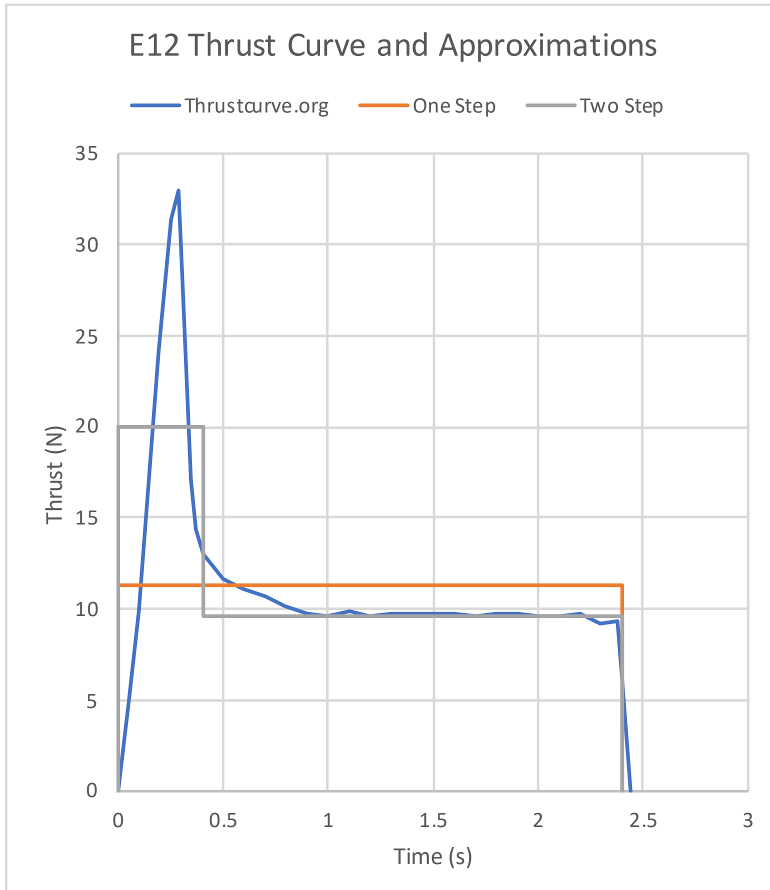
Time (s)	Thrust (N)	Integral
0	0	
0.052	5.045	0.13
0.096	9.91	0.46
0.196	24.144	2.16
0.251	31.351	3.69
0.287	32.973	4.85
0.3	29.91	5.26
0.344	17.117	6.29
0.37	14.414	6.70
0.4	12.973	7.11
0.5	11.712	8.35
0.6	11.171	9.49
0.7	10.631	10.58
0.8	10.09	11.62
0.9	9.73	12.61
1	9.55	13.57
1.101	9.91	14.55
1.2	9.55	15.52
1.3	9.73	16.48
1.4	9.73	17.45
1.5	9.73	18.43
1.6	9.73	19.40
1.7	9.55	20.36
1.8	9.73	21.33
1.9	9.73	22.30
2	9.55	23.26
2.1	9.55	24.22
2.2	9.73	25.18
2.3	9.19	26.13
2.375	9.37	26.83
2.4	5.95	27.02
2.44	0	**27.14**
		11.31

One Step

Time (s)	Thrust (N)
0	0
1E-09	11.31
2.4	11.31
2.4	0

Two Step

Time (s)	Thrust (N)
0	0
1E-09	20
0.4	20
0.4	9.57
2.4	9.57
2.4	0



Thrustcurve.org E9

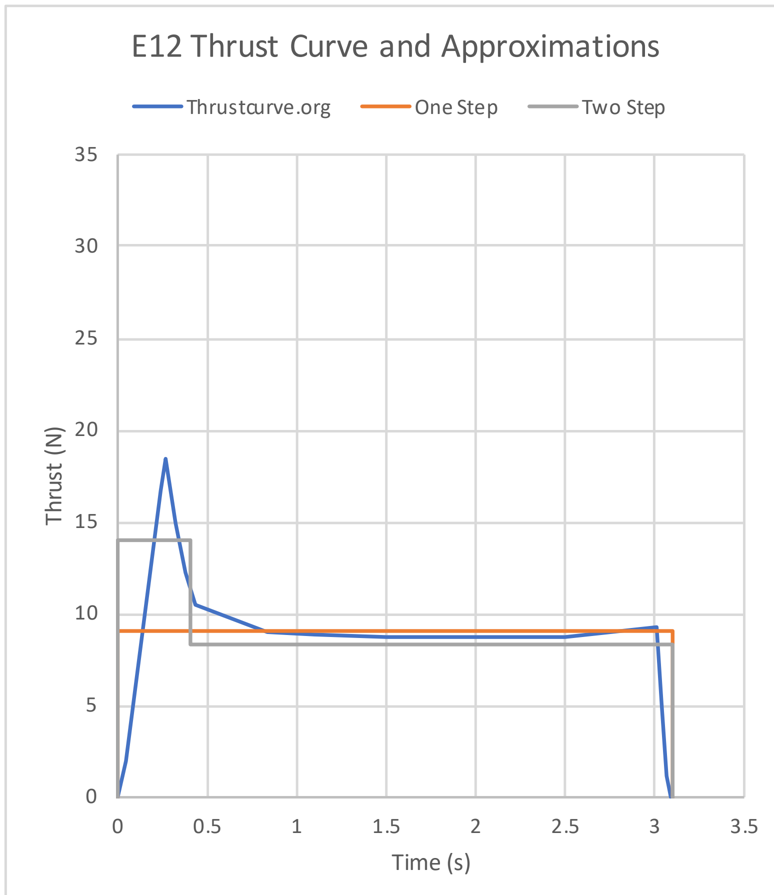
Time (s)	Thrust (N)	Integral
0	0	
0.046	1.913	0.04
0.235	16.696	1.80
0.273	18.435	2.47
0.326	14.957	3.35
0.38	12.174	4.09
0.44	10.435	4.77
0.835	9.043	8.61
1.093	8.87	10.92
1.496	8.696	14.46
1.997	8.696	18.82
2.498	8.696	23.18
3.014	9.217	27.80
3.037	5.043	27.96
3.067	1.217	28.06
3.09	0	**28.07**
		9.05

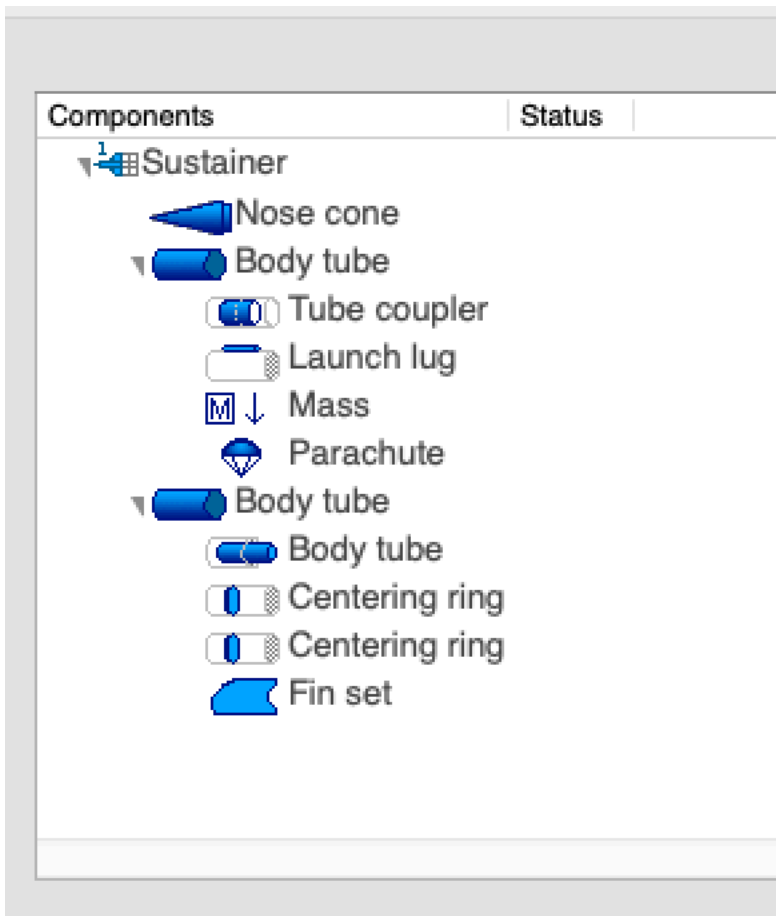
One Step

Time (s)	Thrust (N)
0	0
1E-09	9.05
3.1	9.05
3.1	0

Two Step

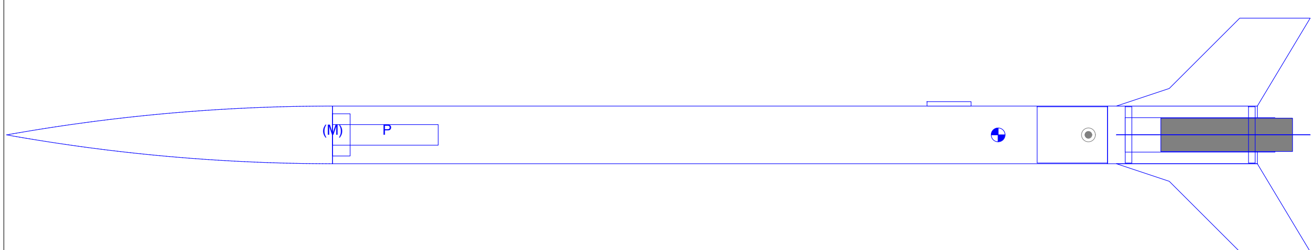
Time (s)	Thrust (N)
0	0
1E-09	14
0.4	14
0.4	8.32
3.1	8.32
3.1	0

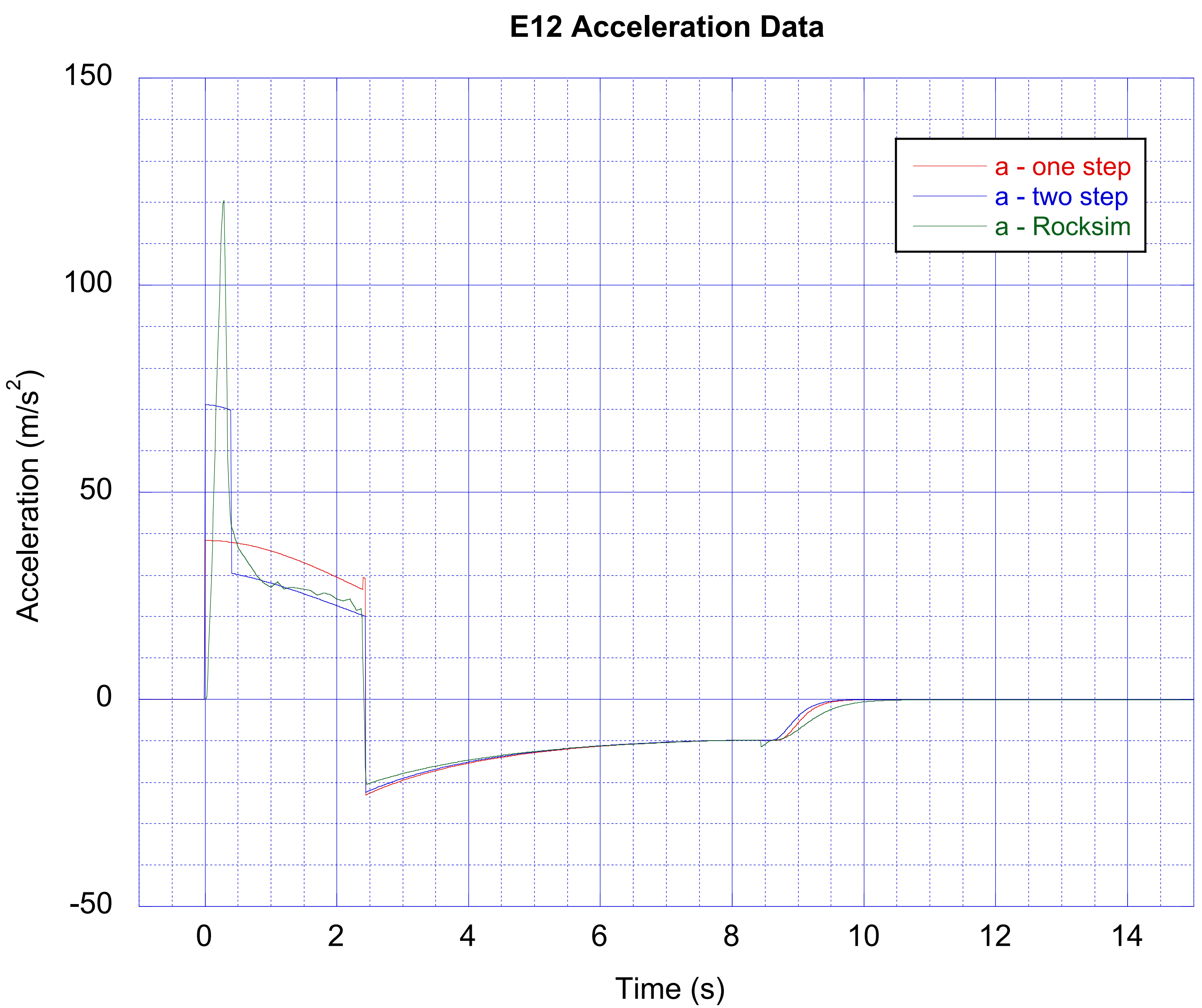


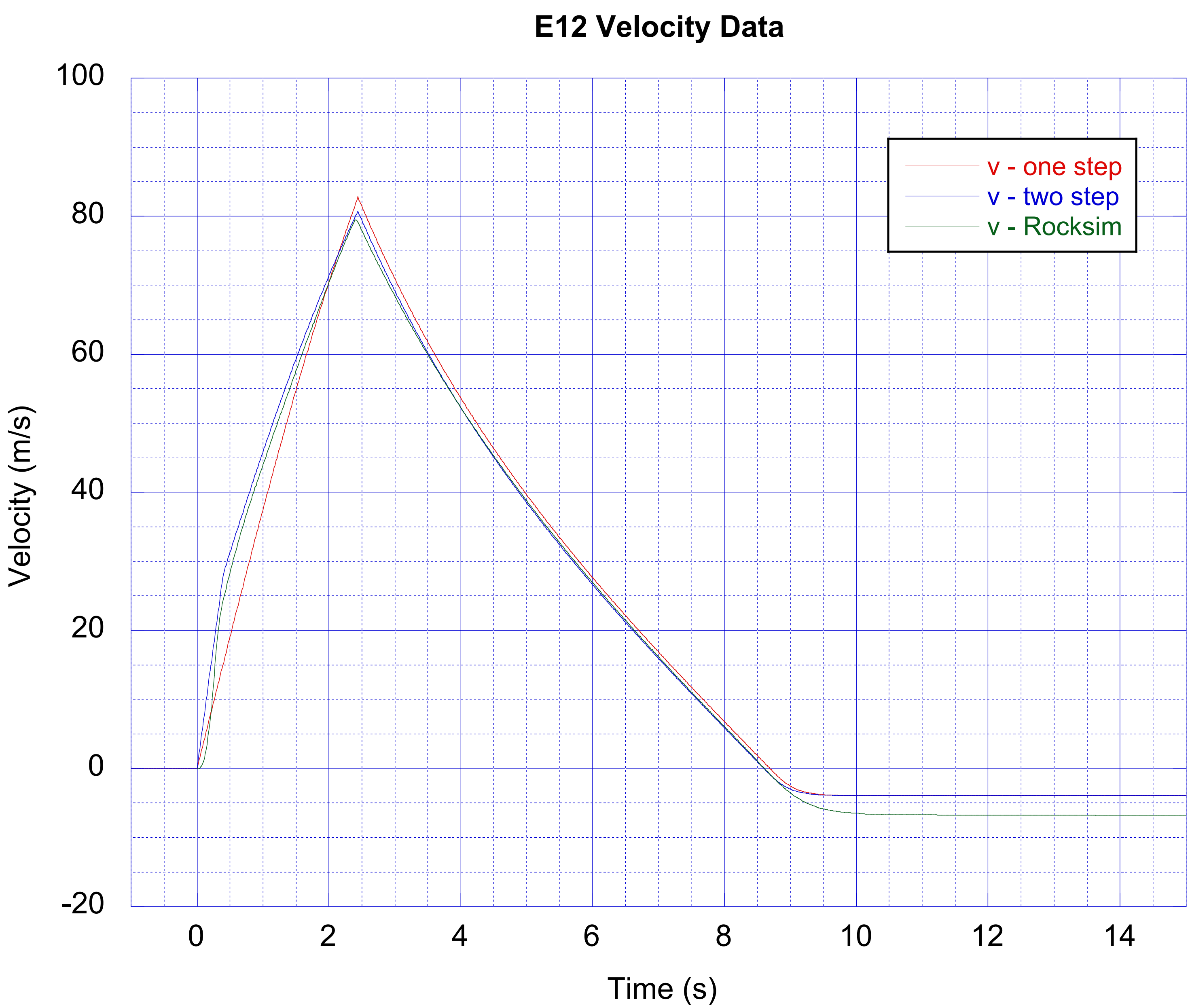


Components
Status
Sustainer
Nose cone
Body tube
Tube coupler
Launch lug
Mass
Parachute
Body tube
Body tube
Centering ring
Centering ring
Fin set

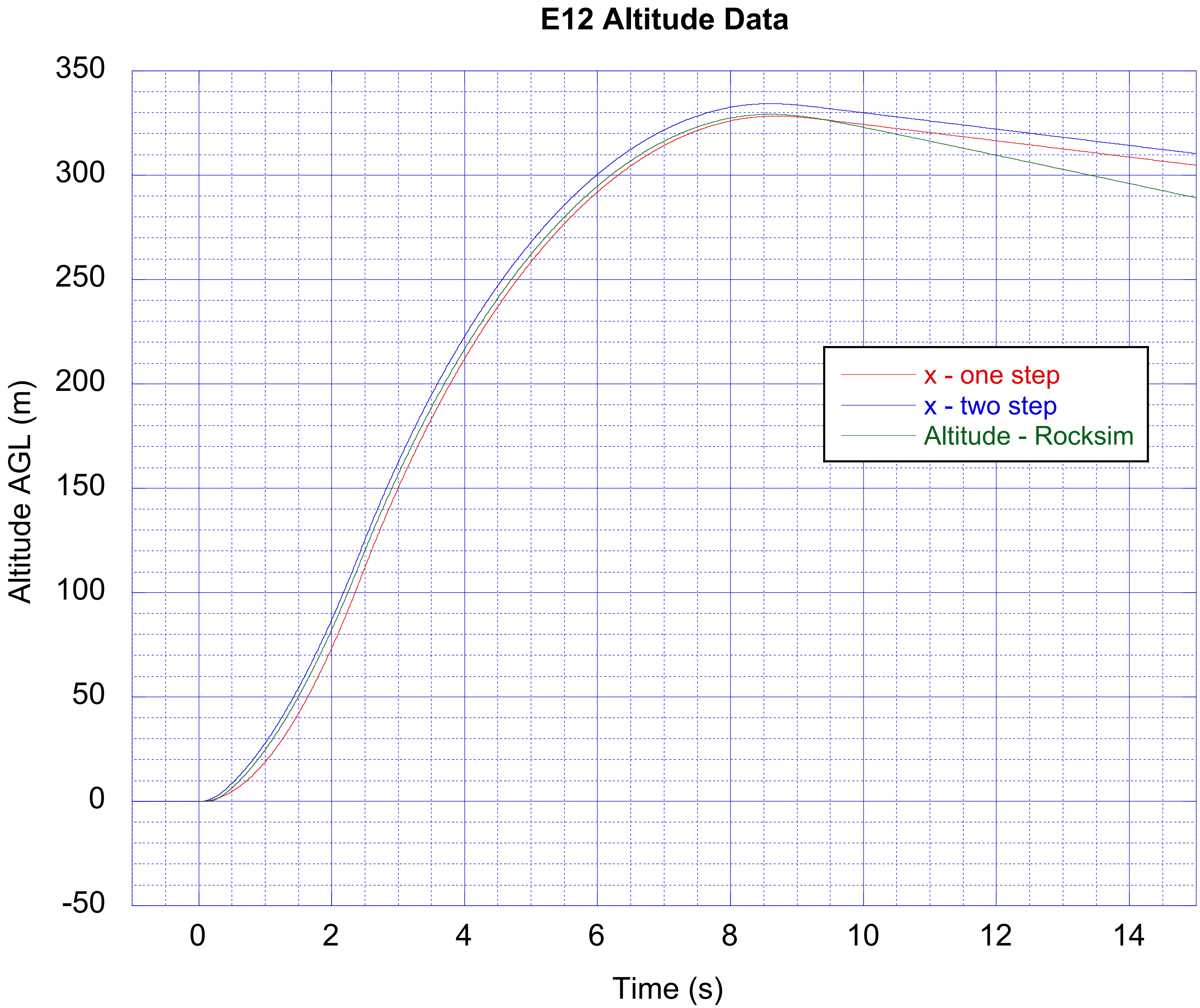
Albee Vector Vehicles Model 01D
Length: 37.0000 In. , Diameter: 1.6400 In. , Span diameter: 6.6400 In.
Mass 260.100 g , Selected stage mass 260.100 g (User specified)
CG: 28.1434 In., CP: 30.7061 In., Margin: 1.56
Engines: [E9-8,]



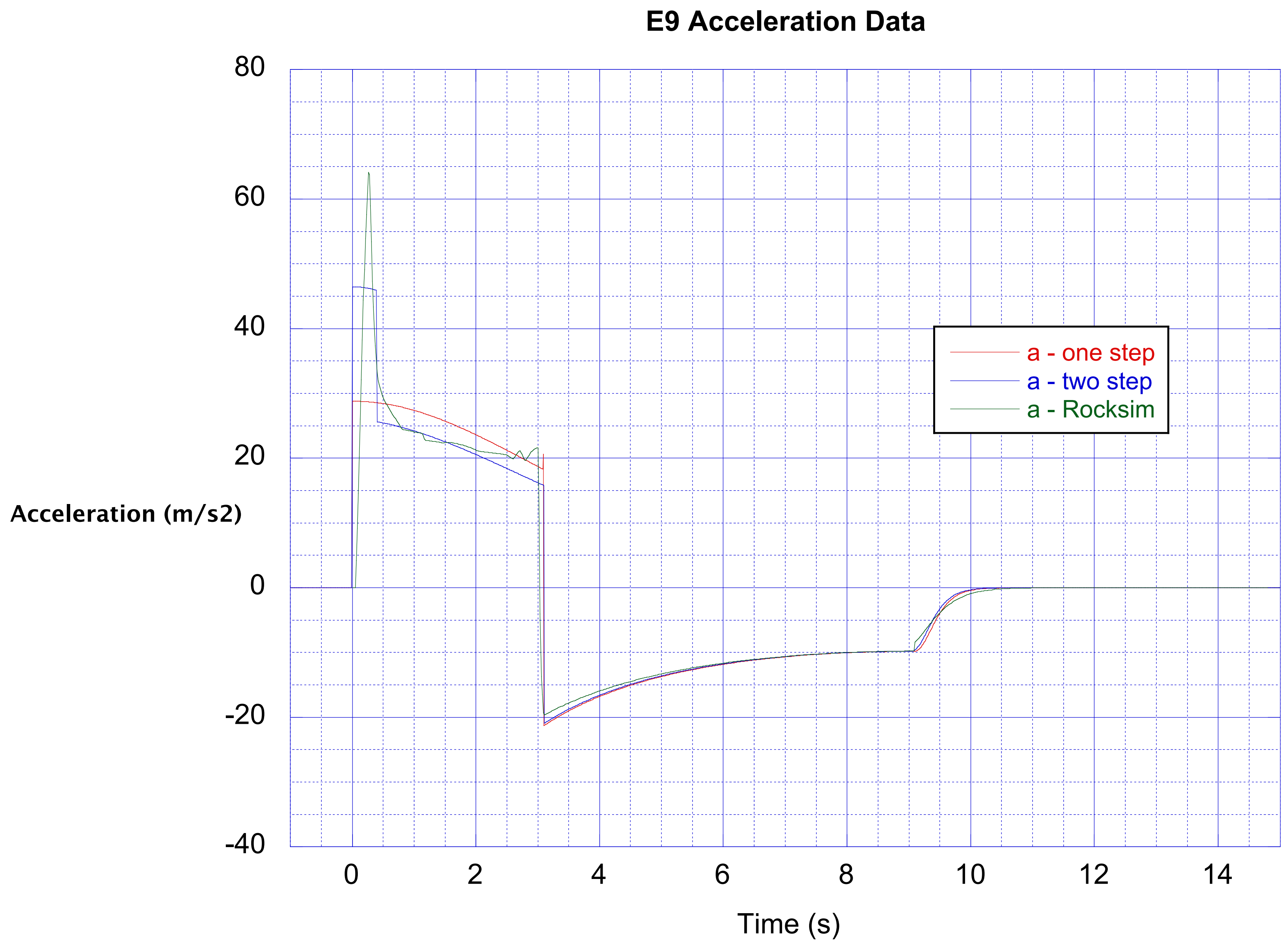

E12 Acceleration Data
a - one step
a - two step
a - Rocksim
Acceleration (m/s^2)
Time (s)
150
100
50
0
-50
0
2
4
6
8
10
12
14



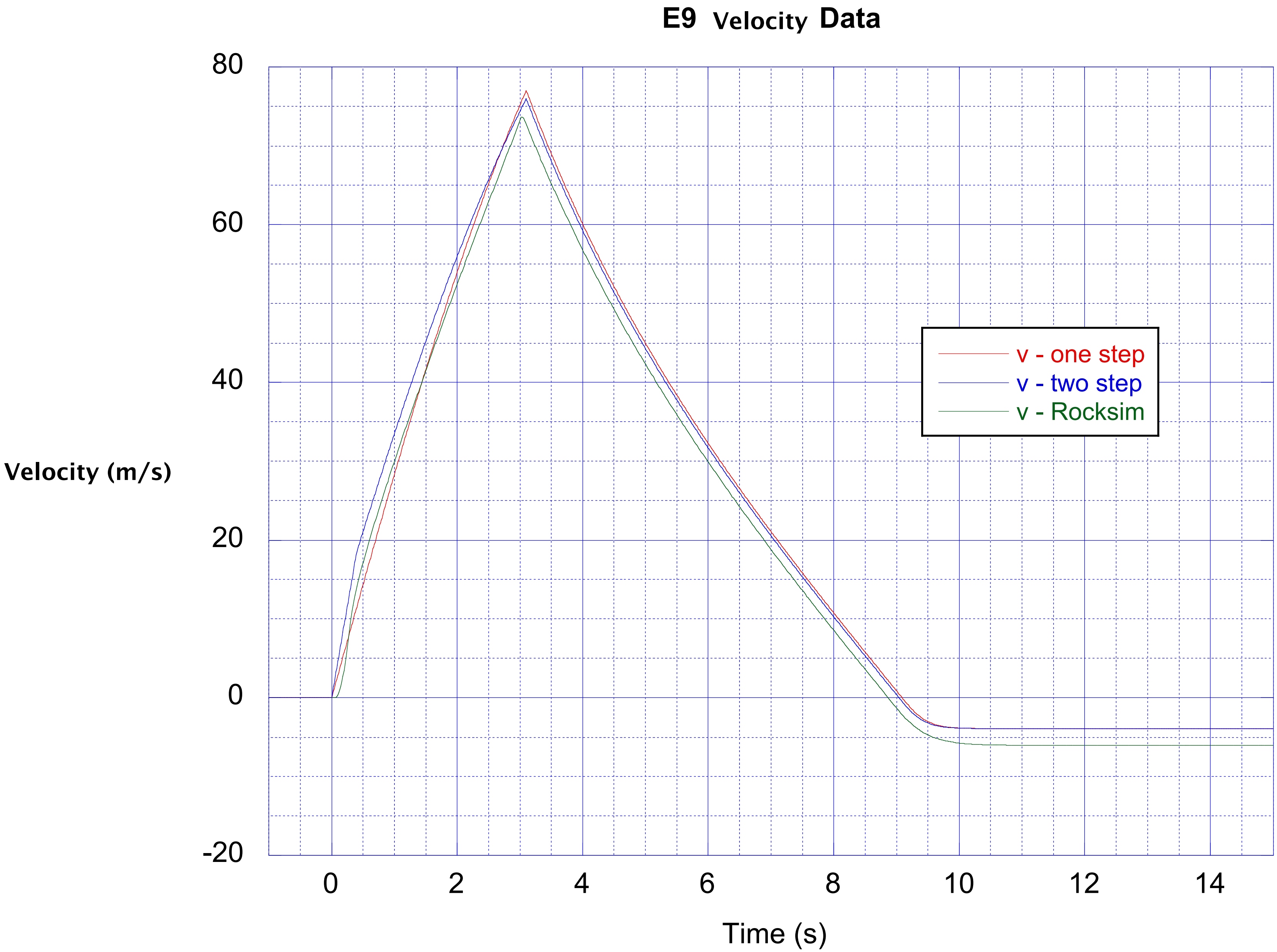
E12 Velocity Data
v - one step
v - two step
v - Rocksim
Velocity (m/s)
Time (s)
100
80
60
40
20
0
-20
0
2
4
6
8
10
12
14

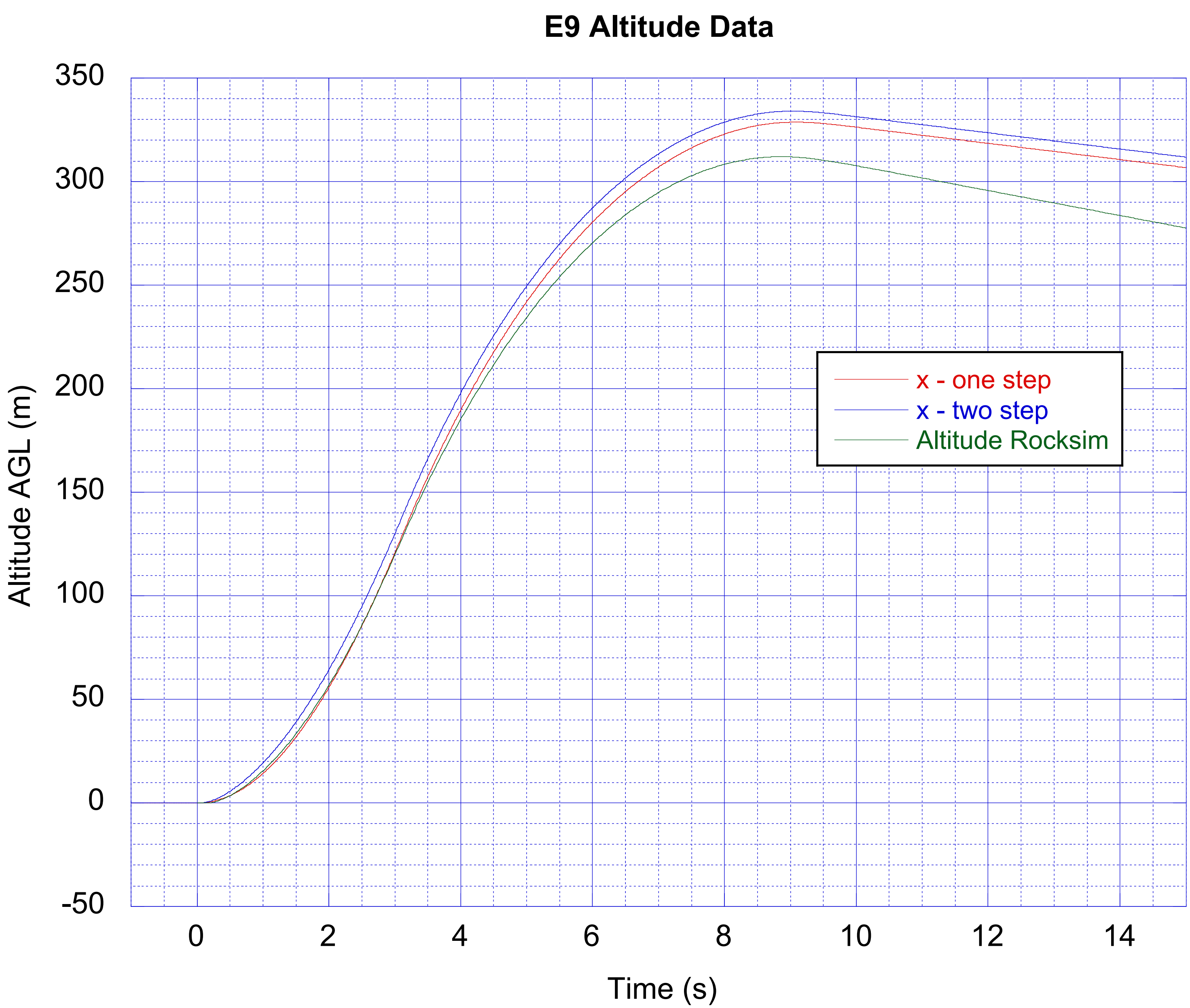


E12 Altitude Data
350
300
250
200
150
100
50
0
-50
Altitude AGL (m)
0
2
4
6
8
10
12
14
Time (s)
x - one step
x - two step
Altitude - Rocksim



E9 Acceleration Data
80
60
40
20
0
-20
-40
Acceleration (m/s2)
0
2
4
6
8
10
12
14
Time (s)
a - one step
a - two step
a - Rocksim


E9 Velocity Data
Velocity (m/s)
80
60
40
20
0
-20
0
2
4
6
8
10
12
14
Time (s)
v - one step
v - two step
v - Rocksim


E9 Altitude Data
350
300
250
200
150
100
50
0
-50
Altitude AGL (m)
0
2
4
6
8
10
12
14
Time (s)
x - one step
x - two step
Altitude Rocksim

E9				E12			
One Step Model		Two Step Model		One Step Model		Two Step Model	
tdelay =	**5.994**	**tdelay =**	**5.9401**	**tdelay =**	**6.2402**	**tdelay =**	**6.0885**
tapogee =	9.094	tapogee =	9.0401	tapogee =	8.6802	tapogee =	8.5285
apogee =	327.8866	apogee =	334.1189	apogee =	326.3895	apogee =	327.0155
timpact =	337.2571	timpact =	343.4355	timpact =	335.3455	timpact =	335.8198
vimpact =	-3.911	vimpact =	-3.911	vimpact =	-3.9019	vimpact =	-3.9019