What Determines the Color We See?

Is color a property of the external world? Are the surfaces in the external world labeled by colors for the eye and brain to see and analyze *passively*? Or is color a property of the brain, which *actively* construct the colors of the surfaces from the information received from the external world? In other words, is color the cause or the consequence of our visual perception? These questions are at the interface of biology and physics, and even of philosiphy. We will address them by answering what determines the color we see. Also, due to these questioins, we will be careful to use the word “light” to indicate the physical signal the eye receives and the word “color” to indicate the corresponding visual perception.

Light – The Electromagnetic Signal

Along the visible band (350 nm – 780 nm) (1 nanometer=10^{-9} meter) of the electromagnetic spectrum, there are only about 128 fully saturated colors that can be distinguished by the human eye. It is the spectral composition, the energy spectral distribution $L(\lambda)$ of the light that is the signal the eye receives for the brain to process. Sometimes $L(\lambda)$ may be referred to as the color, but this is to be distinguished from the actual color we perceive from this light, because the energy spectral distribution is not the only factor that determines the perceived color, as we will discuss later.

Newton’s experiments and contributions (1704):

- Decomposition of white light by a prism into many spectral lights, and recostruction of a white light from the spectral lights by a second prism; A spectral light can not be further decomposed.
- A spinning round plate with sections painted in different colors would look white;
- Realization that perception of color is caused by the spectral composition of the light.

Coherence

If the component waves of a beam of light of a single frequency have a fixed relationship with each other, the light is coherent (e.g., the laser). Two beams of light are coherent if the phase difference between their waves is constant. For example, two beams produced by splitting a single laser beam are coherent and they can form stable interference patterns when combined. On the other hand, a light is noncoherent if its component waves have a random or changing phase relationship, such as light produced by an incandescent lightbulb.

Superposition of Lights

The intensities of incoherent lights add linearly, i.e., the spectral energy distribution of the mixture of two lights is the linear combination of their individual energy distributions, as illustrated in the figure. In other words, the spectral energy distribution of the mixture of a set of n lights with intensities $c_i, \quad (i = 1, \cdots, n)$ is the weighted sum of their individual spectral distributions $L_i(\lambda), \quad (i = 1, \cdots, n)$:

$$L_{mixture}(\lambda) = \sum_{i=1}^{n} c_i L_i(\lambda)$$

The Physics of Light Radiation

Planck (1900) formulated the relationship between the temperature and the spectral energy distribution of a blackbody (a hypothetical body that completely absorbs all radiant energy falling upon it, reaches some equilibrium temperature, and then reemits that energy as quickly as it absorbs it)

$$R(\lambda) = \frac{8\pi hc}{\lambda^5[exp(hc/kT\lambda) - 1]}$$

where c is the speed of light, λ is the wavelength, T is the temperature, h is the Planck constant and k is the Boltzmann constant.

As examples, the energy spectral distribution of several light sources, the direct sun light, the overcast sky, and a tungsten filament lamp are plotted.

Illumination and Reflectance

While an object is illuminated by a light source, what our eyes perceive is the light reflected by the surface of the object:

$$L(\lambda) = R(\lambda)I(\lambda)$$

Obviously $L(\lambda)$ depends on both the illumination $I(\lambda)$ and the reflectance of the object $R(\lambda)$. Typically the illumination has little spatial variation, while the reflectance may have drastical spatial changes representing the details (texture, color, smoothness, etc.) of the object surfaces.

The Perception of Brightness

The link between the physical intensity measurements of the electromagnetic energy and the perceptual brightness of the visible band of the electromagnetic energy is the photopic (bright light mediated by the cones) and scotopic (dim light mediated by the rods) luminnosity function $S(\lambda)$ representing the percived brightness as a function of the wavelength, as shown.

The magnitude of the response of a photoreceptor (the firing rate) is determined by how much light is, or more specifically, how many photons are, absorbed by the photopigment of the photoreceptors. The more light a receptor catches, the stronger its response will be, and the more sensitive the receptor becomes. This property of the photoreceptors can be represented by its photon capture rate (number of photons absorbed per unit time), or its sensitivity, denoted by $S(\lambda)$ as a function of wavelength.

If the light received by the retina is $L(\lambda)$, then the response of a photoreceptor is proportional to the total number of photons absorbed, which can be obtained by

$$\text{Response} = \int S(\lambda)L(\lambda)d\lambda$$

Trichromatic Theory and Color Matching

This theory emerged over the eighteenth century which basically states that any color $L(\lambda)$ can be reproduced by mixing appropriate amounts of three primary colors with energy distributions $P_j(\lambda)$, $\quad (j = 1, 2, 3)$ provided the wavelengths are far enough apart. For exampel, red, green, and blue. As we will see later, "far enough apart" means mathematically the spectral energy distributions of the three colors are linearly independent.

The color matching experiment

The goal of an additive color matching experiment is to superimpose appropriate amounts of the three primaries $P_j \quad (j = 1, 2, 3)$ so that the resulting color L' is perceived the same as a given color L. This can be actually carried out by three projectors that project the three primary colors with adjustable intensities on a screen.

This process is symbolically represented by

$$[L] \equiv [L'] = A_1(L)[P_1] + A_2(L)[P_2]A_3(L)[P_3]$$

This is not a mathematical equation in the normal sense, as the symbols used here have special meanings:

- Anything inside a pair of brackets (e.g., $[L]$ or $[P_j]$) is a color such as a primary color, of certain energy spectral distribution;
- $A_j(L)[P_j]$ denotes some certain amount (represented by $A_j(L)$) of a primary color P_j needed to match a given color L;
- The symbol $+$ represents the actual mixing of the colors by projecting them on the same screen.
- The symbol $\equiv$ means the color on the left-hand side is equivalent to that on the right-hand side in the sense that they are perceived by the human eye as the same color. (But this does not mean that the two energy distributions are necessarily the same, as discussed later.)

As we are usually concerned with the proportions of the color mixing but not the absolute intensities, the color matching process can be normalized by using a reference white color W. The amount of primary P_j needed to match W, represented by $A_j(W)$, is used to normallize the intensity $A_j(L)$ for matching L:

$$T_j(L) = \frac{A_j(L)}{A_j(W)} \quad (j = 1, 2, 3)$$

The $T_j(L)$'s are called the *tristimulus values*.

Negative Tristimulus Values

Sometimes when it is impossible to match a color L by superimposing the given primaries P_j, $\quad (j = 1, 2, 3)$ as described above, it is possible to still match the color in a different way. For example, the mixture of one or two primaries with the given color L may be matched by the mixture of the remaining primaries:

- **Mixing one primary with the given color**

$$[L] + A_1(L)[P_1] \equiv A_2(L)[P_2] + A_3(L)[P_3]$$

with the tristimulus values correspondingly defined as

$$T_1(L) = \frac{-A_1(L)}{A_1(W)} \qquad T_2(L) = \frac{A_2(L)}{A_2(W)} \qquad T_3(L) = \frac{A_3(L)}{A_3(W)}$$

- **Mixing two primaries with the given color**

$$[L] + A_1(L)[P_1] + A_2(L)[P_2] \equiv A_3(L)[P_3]$$

with the tristimulus values correspondingly defined as

$$T_1(L) = \frac{-A_1(L)}{A_1(W)} \qquad T_2(L) = \frac{-A_2(L)}{A_2(W)} \qquad T_3(L) = \frac{A_3(L)}{A_3(W)}$$

Axioms of Color Matching

Based on the observations of many color matching experiments, Grassman (1854) summarized a set of eight axioms:

1. Any color can be matched by a mixture of no more than three colors.
2. A color match at one radiance level holds over a wide range of levels.
3. Components of a mixture of colored lights cannot be resolved by the human eye.
4. The luminance of a colr mixture is equal to the sum of the luminance of its components.
5. Law of additon — if color M matches color N and color P matches color Q, then color M mixed with color P matches color N mixed with color Q:

$$[M] \equiv [N], \;\; [P] \equiv [Q] \quad \Rightarrow \quad [M] + [P] \equiv [N] + [Q]$$

6. Law of subtraction — if the mixture of M and P matches the mixture of N and Q, and if P matches Q, then M matches N:

$$[M] + [P] \equiv [N] + [Q], \quad [P] \equiv [Q], \;\; \Rightarrow \;\; [M] + [N]$$

7. Transitive law — if M matches N and N matches P, then M matches P:

$$[M] + [N], \quad [N] + [P] \quad \Rightarrow \quad [M] \equiv [P]$$

8. Color matching — a given color L can be matched in one of three ways:

   (a)

$$[L] \equiv A_M[M] + A_N[N] + A_P[P]$$

   (b)

$$[L] + A_M[M] \equiv A_N[N] + A_P[P]$$

   (c)

$$[L] + A_M[M] + A_N[N] \equiv A_P[P]$$

The 8th axiom is a summary of the color matching experiments discussed previously. Because the principle of superposition holds in color mixing, we can rewirte a color match

$$[L] \equiv A_1(L)[P_1] + A_2(L)[P_2] + A_3(L)[P_3]$$

as

$$L(\lambda) \equiv A_1(L)P_1(\lambda) + A_2(L)P_2(\lambda) + A_3(L)P_3(\lambda) = \sum_{j=1}^{3} A_j P_j(\lambda)$$

Again, the symbol $\equiv$ represents that the energy spectral distributions on both sides are perceived as the same by the human eye, but as functions of wavelength λ, they are not identical in general.

Comparison — Necessary Condition for Color Vision

- **Monochromatic System**

  If there is only one type of photoreceptors with sensitivity as a function of wavelength $S(\lambda)$, such as the rod cells in the retina, then only one value is produced as the response to a given color $L(\lambda)$:

  $$r = \int S(\lambda)L(\lambda)d\lambda$$

  and the visual system perceives an image of different gray levels. Moreover, it is very likely that some two colors $L(\lambda)$ and $L'(\lambda)$ are responded to by the same value, so long as

  $$r = \int S(\lambda)L(\lambda)d\lambda = r' = \int S(\lambda)L'(\lambda)d\lambda$$

  such as case A of the figure.

- **Dichromatic System**

  The necessary condition for color perception is to have at least two types of photoreceptors with different sensitivity functions $S_i(\lambda) \quad (i = 1, 2)$, so that each color is represented by two values and *comparisons* can be made between the two types of receptors.

  Almost all mammals except primates have only two types of photoreceptors (S type and LM type). But still two colors $L(\lambda)$ and $L'(\lambda)$ may not be distinguished if the two types of receptors respond to the colors in the same way,

  $$r_i = \int S_i(\lambda)L(\lambda)d\lambda = r'_i \int S_i(\lambda)L'(\lambda)d\lambda \quad (i = 1, 2)$$

  such as case B of the figure.

- **Trichromatic System**

  If the visual system has three different types of photoreceptors, then each of the different types of cells respond to a given color $L(\lambda)$ with different intensities:

  $$r_i = \int S_i(\lambda)L(\lambda)d\lambda \quad (i = 1, 2, 3)$$

As the result, more comparisons can be made to distinguish more colors, as shown in case C of the figure. However, can three types of receptors resolve all possible colors?

It is obvious to conclude that the more different types of photoreceptors the visual system has, the more values are available to represent a color, and the more comparisons can be made to distinguish more colors. But how many different types of receptors are necessary to tell *all* possible colors (spectral energy distributions) apart?

Three Types of Cone Cells

To explain the trichromatic theory, Thomas Young (1802) first postulated that at each point in the retina there must exist at least three tiny structures that are sensitive to red, green and violet. This theory was adopted and championed by Hermam von Helmholtz and came to be known as the Young-Helmholtz theory.

This theory was finally confirmed over 150 yeares later by modern physiology (Wald, Brown at Harvard, Marks, Dobelle and MacNichol at Johns Hopkins 1959), in the visual system of the primates, there are three types of cone cells, called the S type, M type and L type, with different sensitivities $S_i(\lambda) \quad (i = 1, 2, 3)$, which peak approximately at 560 nm, 530 nm and 430 nm respectively.

“Red Plus Green Equal to Yellow”

To address the question “can three types of receptors resolve all possible colors?” by an example, we consider three spectral (monochromatic) colors:

- **Red:**

  $$L_r(\lambda) = \delta(\lambda - \lambda_r)$$

  where $\lambda_r = 615$ nm;

- **Yellow:**

  $$L_y(\lambda) = \delta(\lambda - \lambda_y)$$

  where $\lambda_y = 565$ nm;

- **Blue:**

  $$L_g(\lambda) = \delta(\lambda - \lambda_g)$$

  where $\lambda_g = 490$ nm;

As shown in the figure, the responses of the L and M (red and green) cones to color $L_y(\lambda) = \delta(\lambda - \lambda_r)$ are the same as their responses to the mixed color

$$L_{rg}(\lambda) = L_r(\lambda) + L_g(\lambda) = \delta(\lambda - \lambda_r) + \delta(\lambda - \lambda_g)$$

as

$$r_i = \int S_i(\lambda)L_y(\lambda)d\lambda = \int S_i(\lambda)L_{rg}(\lambda)d\lambda \quad (i = 1, 2, 3)$$

So colors $L_y(\lambda)$ and $L_{rg}(\lambda)$ cannot be distinguished by our visual system. That is why “red plus green is yellow”. (In fact, in this particular case, these two colors should still be distinguishable as the S type receptors respond them differently.)

Comparison with Auditory Sensors

If three different types of receptors are not enough to tell *all* possible spectral energy distributions apart, then how many types of receptors are enough? If the entire visible wavelength band (350 nm – 780 nm) is divided into n segments so a color spectral distribution can be approximately specified by n numbers (n degrees of freedom), then obvioulsy we need n receptors. Why don't we have this many receptors?

In our auditory system, we do have many different types of detectors each tuned to respond most optimally a specific narrow range of frequencies. And similarly in our smell and taste sensory systems there are many different types of detectors each for detecting a particular smell or taste. But why don't we have many more types of retina cone cells each for one particular band of wavelength?

The answer to this question is obvious if we realize that vision takes place in a two-dimensional visual field (the surface area of the retina). In order to be able to sense the color of every position in the field, there would have to be many such receptors, too many to be possibly contained in the retina of limited area. Moreover, the brain would also have to processing much more visual information coming from the receptors in the eye.

But we do perceive many more different colors than the three typse of detectors red, green and blue. All these different colors are represented internally in our brain as different ratios between the three types of sensors or neurons tuned for the three primary colors. In orther words, there are not "grandmother cells" each for a different color.

“Yellow Plus Blue Equal to .... ?”

Almost all children know the mixture of blue and yellow paints looks green. But why? Does this also mean that the mixture of blue light and yellow light also looks green? If not, what color should the mixture of light looks like and why?

Helmholz was the first to explain why the mixture of yellow and blue paints looked green. To answer these questions, we should first understand why paints have different colors. A red paint on a white paper does not reflect extra red light than the paper without the red paint. Instead, the red paint *absorbs* all other colors than red. In general, the paints act as band-pass filters only allowing frequencies in a certain range to pass. On the other hand, a red light has most of its energy distributed in the frequency band of red light. In other words, the spectral distribution of a mixture of lights is the sum of the spectral distributions of the individual lights (a logic OR), while the spectral distribution of a mixture of paints is the intersection of the distributions of the individual paints (a logic AND). As a simple example, we easily see that the mixture of paints of all colors looks balck, while the mixture of lights of all colors looks white.

Going back to the yellow plus blue problem, if we are mixing paints, the intersection of the spectral distributions (assuming they are wide enough) is in the frequency range for green. But if we are mixing lights, then the sum of the two distributions, if wide enough, may be as wide as to cover most of the visible visible frequency range so that the mixture may look light gray.

In the previous red-plus-green case, if we were mixing paints instead of lights, the mixture may look dark bluish, as both the red and green components of the white light are absorbed.

Trichromatic Theory

Now we can revisit the color matching issue, and discuss the real meaning of the equivalent symbol "≡".

The visual system and the color system are described by

- **The Visual System** The three types of photoreceptors with senstivities $S_i(\lambda), \quad (i = 1, 2, 3)$;
- **The Color System** The three primaries with energy spectral distributions $P_j(\lambda), \quad (j = 1, 2, 3)$ together with the reference white $W(\lambda)$.

We want to match a given color $L(\lambda)$ with another color produced by mixing the three primaries

$$L'(\lambda) = \sum_{j=1}^{3} A_j(L) P_j(\lambda)$$

Since color matching now means specifically the three types of photoreceptors have the same responses to the two colors, we must require

$$\begin{aligned} & r_i(L) = \int L(\lambda) S_i(\lambda) d\lambda \\ = \quad & r_i(L') = \int L'(\lambda) S_i(\lambda) d\lambda = \int [\sum_{j=1}^{3} A_j(L) P_j(\lambda)] S_i(\lambda) d\lambda \\ = \quad & \sum_{j=1}^{3} A_j(L) \int S_i(\lambda) P_j(\lambda) d\lambda = \sum_{j=1}^{3} A_j(L) a_{ij} \quad (i = 1, 2, 3) \end{aligned}$$

where a_{ij} is defined as the response of the ith cells to the jth primary:

$$a_{ij} \stackrel{\triangle}{=} r_i(P_j) = \int S_i(\lambda) P_j(\lambda) d\lambda \qquad (i, j = 1, 2, 3)$$

Rewriting these equations as

$$\sum_{j=1}^{3} a_{ij} A_j(L) = \int S_i(\lambda) L(\lambda) d\lambda \quad (i = 1, 2, 3)$$

we get the *color matching equations* which can be solved to get $A_j(L)$, $(i = 1, 2, 3)$, the weights for mixing the three primaries. The resulting color L' will be perceived the same as the given color L:

$$L(\lambda) \equiv L'(\lambda) = \sum_{j=1}^{3} A_j(L) P_j(\lambda)$$

although they in general have very different energy spectral distributions

$$L(\lambda) \neq L'(\lambda) = \sum_{j=1}^{3} A_j(L) P_j(\lambda)$$

In general if the three types of photoreceptors' responses to two colors $L_1(\lambda)$ and $L_2(\lambda)$ are exactly the same

$$r_i(L_1) = \int L_1(\lambda) S_i(\lambda) d\lambda = r_i(L_2) = \int L_2(\lambda) S_i(\lambda) d\lambda \quad (i = 1, 2, 3)$$

then the two colors are indistinguishable and are called *metamers*. There are infinitely many possible metameric spectral energy distributions that cause the same responses in the visual system.

Calibration with a Reference White

The magnitudes of the weights $A_j(L)$ will be directly affected by the intensity of the given color $L(\lambda)$. As we are mostly concerned with the proportions of the three primaries instead of the absolute intensity, we want to normalize $A_i(L)$'s by calibrating them against a reference white light source $W(\lambda)$.

As discussed before, we define the *tristimulus values* of a given color L as

$$T_j(L) = \frac{A_j(L)}{A_j(W)} \qquad (j = 1, 2, 3)$$

where $A_j(W)$'s is the weights for matching the reference white $W(\lambda)$ using the three primaries, which can be obtained by solving these matching equations for the reference white:

$$\sum_{j=1}^{3} a_{ij} A_j(W) = \int S_i(\lambda) W(\lambda) d\lambda \quad (i = 1, 2, 3)$$

In particular, the tristimulus values for the reference white itself is always 1:

$$T_j(L)|_{L=W} = A_j(L)/A_j(W)|_{L=W} = A_j(W)/A_j(W) = 1$$

Now the color matching equations for an arbitrary color $L(\lambda)$ become

$$\sum_{j=1}^{3} a_{ij} T_j(L) A_j(W) = \int S_i(\lambda) L(\lambda) d\lambda \quad (i = 1, 2, 3)$$

Solving these equations, we can get $T_j(L), \quad (j = 1, 2, 3)$ for matching the given color $L(\lambda)$

Spectral matching curves

If the color to be matched is a spectral color of a single wavelength λ' with unit energy:

$$L(\lambda) = \delta(\lambda - \lambda')$$

the matching equations become

$$\sum_{j=1}^{3} A_j(W)T_j(\lambda')a_{ij} = \int S_i(\lambda)\delta(\lambda - \lambda')d\lambda = S_i(\lambda') \quad (i = 1, 2, 3)$$

Solving these three simultaneous equations we can get the tristimulus values $T_j(\lambda') \quad (i = 1, 2, 3)$ for the given spectral color $\delta(\lambda - \lambda')$. As functions of wavelength λ, these $T_j(\lambda)$'s are called the *spectral matching curves* representing amount of each primary needed to match a flat spectral distribution.

The spectral matching curves can also be used to obtain tristimulus values $T_j(L)$ for any given color $L(\lambda)$. To see this, we substitute

$$S_i(\lambda) = \sum_{j=1}^{3} A_j(W)T_j(\lambda)a_{ij}$$

into the color matching equations to get

$$\sum_{j=1}^{3} a_{ij}A_j(W)T_j(L) = \int S_i(\lambda)L(\lambda)d\lambda = \sum_{j=1}^{3} A_j(W)a_{ij} \int T_j(\lambda)L(\lambda)d\lambda$$

As this equation holds for all three types of cells (i=1,2,3), we see that

$$T_j(L) = \int L(\lambda)T_j(\lambda)d\lambda \quad (j = 1, 2, 3)$$

the tristimulus values for matching any color $L(\lambda)$ can be easily obtained from the spectral matching curves $T_j(\lambda)$.

CIE's Spectral RGB Primaries

As an example, we want to find the spectral matching curves using a particular set of primaries composed of monochromatic (spectral) red, green blue, as recommended by the Commission Internationale de l'Eclairage (CIE):

- Red: $P_1(\lambda) = \delta(\lambda - \lambda_1)$, where 5 $\lambda_1 = 700$ nm;
- Green: $P_2(\lambda) = \delta(\lambda - \lambda_2)$, where $\lambda_2 = 546.1$ nm;
- Blue: $P_3(\lambda) = \delta(\lambda - \lambda_3)$, where $\lambda_3 = 435.8$ nm.

and the CIE reference white defined by a flat spectral distribution:

$$W(\lambda) = 1$$

To find the spectral matching curves, we first find a_{ij}, the response of the ith type of cones to the jth primary:

$$a_{ij} = r_i(P_j) = \int S_i(\lambda)\delta(\lambda - \lambda_j)d\lambda = S_i(\lambda_j)$$

We next find the weights $A_j(W)$ to match the CIE reference white by solving the color matching equations for the reference white

$$\sum_{j=1}^{3} A_j(W)T_j(W)a_{ij} = \int S_i(\lambda)W(\lambda)d\lambda = \int S_i(\lambda)d\lambda$$

where $T_j(W) = 1$, as noted before.

Now we substitute these $A_j(W)$ and $a_{ij} = S_i(\lambda_j)$ into the matching equations for a spectral color $\delta(\lambda - \lambda')$ to get

$$\sum_{j=1}^{3} A_j(W)T_j(\lambda)a_{ij} = \sum_{j=1}^{3} A_j(W)T_j(\lambda)S_i(\lambda_j) = S_i(\lambda) \quad (i = 1, 2, 3)$$

Solving these equations we get $T_j(\lambda) \quad (j = 1, 2, 3)$ for any wavelength λ, i.e., the three spectral matching curves representing the amount of each of the spectral red, green and blue needed to match each spectral color, as shown in the plot. Note that the interpretation of the negative values for spectral red at certain wavelengths ($435.8nm < \lambda < 546.1nm$) is the same as discussed before. Namely, spectral colors of wavelength in this range should be mixed with the spectral red to be matched with the mixture of the spectral green and spectral blue.

The Color Space

Based on the tristimulus theroy, any given color $L(\lambda)$ can be matched by mixing proper amounts of three primaries:

$$[L(\lambda)] \equiv \sum_{j=1}^{3} A_j(L)[P_j(\lambda)]$$

Because the intensities of incoherent lights add linearly, i.e., mixing colors L_1 and the L_2 results a linear combination (weighted sum) of the two colors, we can use a three dimensional color space spanned by a set of three primary colors to represent all colors. In this space, each point, or a vector, is a color. And a color L produced by mixing two other colors L_1 and L_2 is represented by the vector sum of the vectors for L_1 and L_2.

Any three linearly independent colors can be considered as the primaries $P_j(\lambda), \;\; (j = 1, 2, 3)$ and used as the three bases of the 3D color space. Here linearly independent means none of the three primaries can be written as the linear combination of the other two, i.e.,

$$a_1 P_1(\lambda) + a_2 P_2(\lambda) + a_3 P_3(\lambda) \neq 0$$

unless $a_1 = a_2 = a_3 = 0$.

Any given color L in the color space can be specified by its coordiates $\{A_1(L),\; A_2(L),\; A_3(L)\}$, just like a point in a 3D Cartesian coordinate system can be represented by it coordinates $\{x, y, z\}$ with the following differences

- the color space is a space of functions of λ, instead of a Cartesian space of geometric points or vectors;
- orthogonal axes are used in Cartesian coordinate system, while axes used in the color space are usually not orthogonal;
- as a result of the above, different orthogonal axes are related by *orthogonal transformations* in Cartesian coordinate system while the axes in the color space are related by *projective transformation.*

Question:

As we know an arbitrary light $L(\lambda)$ is just a function of wavelength λ. By claiming all lights can be represented in a 3D color space spanned by a set of three primaries, are we also claiming that all possible functions of λ can be represented as a linear combination of three primaries (linearly independent) functions $P_i(\lambda)$ $(i = 1, 2, 3)$, an obvious wrong mathematical statement?

Fourier expansion...

Answer:

Given the sensitivity functions $S_i(\lambda)$ of the three photoreceptors of a particular visual system, all possible lights as functions of wavelength $L(\lambda)$ are mapped to a 3D color space. Each point in this space represents a set of infinite number of all possible lights, the "matching colors", in the sense that they are responded to by the three types of photoreceptors in the visual system in exactly the same way. In other words, each point in the color space is a family of metamers. Now we see one of the reasons why the words light and color should be distinguished.

LMS Coordinate System

The response intensities of the L, M and S cones can be used as the bases of the 3D color space. Any *perceived* color is represented by a ray initiated from the origin along a certain direction determined by the proportions between the LMS responses. Note that every point in this 3D space represents a unique perceived color, but not a unique physical color of unique energy spectral distributions. In other words, a point in the color space represents all energy spectral distributions that are perceived the same.

All spectral (monochromatic) color of single wavelength ranging from 350 nm to 780 nm form a curved surface like a bent fan starting from the long wavelength at 780 nm very close to the L axis and ending at short wavelength at 350 very close to the S axis. This surface is everywhere convex. If this surface is truncated by a specific intensity the resulting edge is the *spectrum locus*. The line connecting the two end points (for red and violet) is called the *purple line*.

Since any color as a function of the wavelength λ is a linear combination of all the spectral colors,

$$L(\lambda) = \int c(\lambda')\delta(\lambda - \lambda')d\lambda'$$

it is represented in the color space as some vector sum of the vectors along the spectral surface. We conclude that all perceivable colors are inside the conical solid enclosed by the spectral surface and the plane formed by the purple line and the origin. The cross section of this solid and the plane representing some certain intensity is called the *chromaticity diagram* representing all visible colors of the particular intensity. Usually this cross section in 3D color space is projected onto a 2D subspace spanned by two basis functions defined in some specific way (e.g., red and green).

CIE's Spectral RGB Coordinate System

The CIE spectral RGB discussed before can also be used as the bases of the color space. Each color is a point in the space with the coordiates representing the amounts of the RGB primaries needed to match the color. We see some colors have negative Red coordinates as they need to be matched with negative Red primary. Specifically, the spectral colors in the range of $435.8nm < \lambda < 546.1nm$ are on the negative side of red axis, as indicated by the intersections of the spectrum locus with the vertical (green) axis. In other words, only the colors in the positive octant can be physically produced by mixing these three primaries.

CIE's XYZ Coordinate System

CIE developed a set of three *hypothetical* (unreal, imaginary) primaries X, Y, and Z in ordor to be able to match *all* colors by mixing these primaries with *positive* weights. This means that in the color space the three axes of X, Y and Z must lie outside the conical solid first discussed in the LMS coordinate system. (Or, equivalently the conical solid must lie entirely inside the positive octant of the XYZ coordinate system.) The XYZ primaries are not real because they cannot be physically realized. Otherwise they could produce colors outside the conical solid representing *all* possible colors. The tristimulus spectral matching curves for these XYZ primaries are shown in the figure, and indeed they are always positive for the entire range of visible wavelengths (and therefore all linear combinations thereof).

RGB Primaries of Color Monitors

The colors on TV and computer screens are produced by three types of phosphors coated on the inner surface of the screen each actived by an electronic beam. When actived by the electronic beam, the phosphors emit respectively red, green, and blue light with certain spectral distributions and with intensity proportioinal to the current. By mixing these three colors as the primaries with different proportions, many other colors can be produced. These three primaries can be used as the basis functions in the color space as shown. As these primaries are real and actually implemented, the positive parts of their corresonding axes must lie entirely inside the conical solid representing all visible colors. In other words, the colors producable by the phosphors are only a subset of all visible colors. In the figure, the NTSC (National Television SYstems Committee) receiver primaries are shown.

Question:

Is it possible to construct a set of three **realizable** primaries which can be used to match all perceivable colors with positive tristimulus values (weights)?

Answer:

Any set of three primaries is always represented by some three straight lines in the 3D color space, and all colors matchable by these primaries are inside the subspace spanned by the positive part of the three straight lines. But, on the other hand, all perceivable colors are inside the conical solid with curved surface (the spectrum locus and the purple line) which cannot be fit precisely by any three straight lines. As the result, we can either try to find a set of primaries with the positive part of the corresponding straight lines completely inside the conical solid, but still covering as much space as possible (such as the RGB primaries used for color screens) to make sure positive weights; or use a set of *unrealizable* primaries which can lie outside the conical solid so that the solid can be enclosed completely in the space spanned by the positive part of the corresponding straight lines, such as the CIE XYZ primaries. In other words, it is *impossible* to match all perceivable colors by a set of realizable primaries.

Transformation between Different Color Systems

As we have seen, any three linearly independent colors can be considered as the primary colors and used as the bases of the 3D color space. Therefore the same color may be represented by different tristimulus values under different color systems of different primaries. For example, consider color $L(\lambda)$ matched in the color space by two sets of primaries $P_j(\lambda)\ \ (j=1,2,3)$ and $Q_j(\lambda)\ \ (j=1,2,3)$:

$$L(\lambda) \equiv \sum_{j=1}^{3} A_j P_j(\lambda) \equiv \sum_{j=1}^{3} B_j Q_j(\lambda)$$

As all terms in this equation are matching colors, they must all cause the same receptor responses:

$$r_i = \int L(\lambda) S_i(\lambda) d\lambda = \sum_{j=1}^{3} A_j \int P_j(\lambda) S_i(\lambda) d\lambda = \sum_{j=1}^{3} B_j \int Q_j(\lambda) S_i(\lambda) d\lambda \quad (i=1,2,3)$$

To find the relationship between the two sets of tristimulus values $[A_1, A_2, A_3]$ and $[B_1, B_2, B_3]$, we first consider matching the second set of primaries by the first set:

$$Q_k(\lambda) \equiv Q'_k(\lambda) = \sum_{j=1}^{3} C_{kj} P_j(\lambda) \quad (k=1,2,3)$$

Note that in general $Q'_k(\lambda)$ and $Q_k(\lambda)$ do not have identical spectral energy distributions, but they are matching colors represented by the same point in the 3D color space. Now we can use $Q'_k(\lambda)$'s as the primaries to match the color $L(\lambda)$ so that

$$\begin{aligned} r_i &= \sum_{k=1}^{3} B_k \int Q'_k(\lambda) S_i(\lambda) d\lambda \\ &= \sum_{k=1}^{3} B_k \int \sum_{j=1}^{3} C_{kj} P_j(\lambda) S_i(\lambda) d\lambda \\ &= \sum_{k=1}^{3} B_k \sum_{j=1}^{3} C_{kj} \int P_j(\lambda) S_i(\lambda) d\lambda \\ &= \sum_{j=1}^{3} A_j \int P_j(\lambda) S_i(\lambda) d\lambda \quad (i=1,2,3) \end{aligned}$$

The last equation is due to the fact that this color is also matched by the primary $P_j(\lambda)$'s as assumed originally, and can be written as

$$\sum_{j=1}^{3}[A_j - \sum_{k=1}^{3} C_{kj} B_k] \int P_j(\lambda) S_i(\lambda) d\lambda \quad (i = 1, 2, 3)$$

As in general

$$\int P_j(\lambda) S_i(\lambda) d\lambda \neq 0$$

we must have

$$A_j = \sum_{k=1}^{3} C_{kj} B_k \quad (j = 1, 2, 3)$$

Now the relationship between the two sets of primaries can be summarized as

- For the primaries:

$$Q_k = \sum j = 1^3 C_{kj} P_j \quad (k = 1, 2, 3)$$

- For the tristimulus values:

$$A_j = \sum k = 1^3 C_{kj} B_k \quad (j = 1, 2, 3)$$

To express the above relationship in matrix form, define **P**, **Q**, **A** and **B**: as 3D vectors

$$\mathbf{P} \triangleq [P_1, P_2, P_3]^T$$

$$\mathbf{Q} \triangleq [Q_1, Q_2, Q_3]^T$$

$$\mathbf{A} \triangleq [A_1, A_2, A_3]^T$$

$$\mathbf{B} \triangleq [B_1, B_2, B_3]^T$$

and C as a 3 by 3 matrix

$$\mathbf{C} \triangleq \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

and we have

$$\mathbf{Q} = \mathbf{C}\mathbf{P}$$

and

$$\mathbf{A} = \mathbf{C}^T\mathbf{B}$$

where $\mathbf{C}^T$ represents the transpose of matrix $\mathbf{C}$. These linear relations are called *projective transforms*.

CIE's XYZ Primaries

The Commission Internationale de l'Eclairage (CIE) defined three standard primaries called **X**, **Y**, and **Z**. Any color $L(\lambda)$ can be matched using these primaries with positive weights $X(L)$, $Y(L)$ and $Z(L)$.

The *chromaticity* values of a color is defined by its weights for the three primaries normalized by the total energy $X+Y+Z$:

$$x = \frac{X}{X+Y+Z}, \quad y = \frac{Y}{X+Y+Z}, \quad z = \frac{Z}{X+Y+Z}$$

so that $x+y+z=1$. Chromaticity values depend on the hue and saturation of the color, but are independent of the intensity.

All visible colors are represented by the points inside an enclosed area in the $X+Y+Z=1$ plane. And the chromaticity diagram is the projection of this enclosed area on (X,Y) plane.

Three Components of Color

As we now know all perceived colors can be represented by three independent variables, either the LMS responses, or the tristimulus values under the primaries of a given color system, such as RGB or XYZ. In many situations (e.g., computer image processing), it is more convenient to represent a color by a different set of three independent variables HSL (or HSI, HSV). These are defined as

- *Hue:* the dominant wavelength, the redness of red, greenness of green, etc.
- *Saturation:* the purity of the color, or how much white is contained in the color. For example, red and royal blue are more saturated than pink and sky blue, respectively.
- *Luminance (intensity, value):* the intensity of the light.

  Brightness: for measuring self-luminous objects that emits light (CRT, etc.)

  Lightness: for measuring reflected light.

The hue, saturation and luminance can be obtained given the tristimulus values of a color under any color system.

RGB cube, RGB cone, etc. ...