B^{-1}

A

$y = f(x, \theta)$

$\theta = [\theta_1, \cdots, \theta_N]^T$

$$D = \{(x_n, y_n), \ \ n = 1, \cdots, N\}$$

θ

D

$$J(\theta) = \sum_{n=1}^{N} [y_n - f((x_n, \theta))]^2$$

$$J(\theta) = L(\theta|D) \propto p(D|\theta) = \prod_{n=1}^{N} p(y_n|x_n, \theta)$$

θ^*

$$y = f(x) = f(x_1, \cdots, x_N)$$

R^N

$\{x_1, \cdots, x_N\}$

$x^* = [x_1^2, \cdots, x_N^*]^T$

$f(x)$

$-f(x)$

$$x^* =_x f(x), \qquad f(x^*) = \min_x f(x)$$

x

$g_f(x)$

$$g_f(x) = \bigtriangledown_x f(x) = \frac{df(x)}{dx} = \begin{bmatrix} \partial f / \partial x_1 \\ \\ \partial f / \partial x_N \end{bmatrix} = 0$$

x^*

$f(x)$

$$x_{n+1} = x_n - J_f^{-1}(x_n)\, f(x_n)$$

$g_f(x) = 0$

$J_g(x) = H_f(x)$

$$x_{n+1} = x_n - J_g^{-1}(x_n)\, g_f(x_n) = x_n - H_f^{-1}(x_n)\, g_f(x_n)$$

g_f

$J_g = H_f$

$$f(x) = [f_1(x), \cdots, f_N(x)]^T = 0$$

$$J(x) = \frac{1}{2} f^T(x) f(x) = \frac{1}{2} ||f(x)||^2 = \frac{1}{2} \sum_{i=1}^{N} |f_i(x)|^2$$

$J(x)$

$$g_J(x) = \bigtriangledown_x J(x) = \frac{d}{dx}\left[\frac{1}{2}f^T(x)f(x)\right] = f'(x)\ \ f(x) = J_f(x)\,f(x) = 0$$

$J_f(x) = f'(x)$

$f(x) = 0$

$f(x)$

x_0

$f(x)$

=

$$f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n + \cdots$$

≈

$$f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 = q(x)$$

$f'(x)$

$f''(x)$

$q(x)$

$f(x) = q(x)$

$q'(x)$

$$\frac{d}{dx}q(x) = \frac{d}{dx}\left[f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2\right]$$

$$f'(x_0) + f''(x_0)(x - x_0) = 0$$

$$x^* = x = x_0 - \frac{f'(x_0)}{f''(x_0)} = x_0 + \Delta x_0$$

$\Delta x_0 = -f'(x_0)/f''(x_0)$

x^*

$f(x^*)$

$f''(x^*) > 0$

$f''(x^*) < 0$

$f(x) \neq q(x)$

$$x_{n+1} = x_n + \Delta x_n = x_n - \frac{f'(x_n)}{f''(x_n)}, \qquad n = 0,\ 1,\ 2, \cdots$$

$\Delta x_n = -f'(x_n)/f''(x_n)$

x_n

$q(x_n)$

x_{n+1}

$q(x_{n+1})$

$n \to \infty$

$x_{n+1} = x_n - f'(x_n)/f''(x_n)$

$f'(x) = 0$

$$f(x) = f(x_1, \cdots, x_N)$$

$q(x)$

x_0

$$f(x) \approx f(x_0) + g_0^T (x - x_0) + \frac{1}{2}(x - x_0)^T H_0 \, (x - x_0) = q(x)$$

g_0

H_0

g_0

$$g_f(x_0) = \frac{d}{dx} f(x_0) = \begin{bmatrix} \frac{\partial f(x_0)}{\partial x_1} \\ \\ \frac{\partial f(x_0)}{\partial x_N} \end{bmatrix},$$

H_0

$$H_f(x_0) = \frac{d}{dx} g(x_0) = \frac{d^2}{dx^2} f(x_0) = \begin{bmatrix} \frac{\partial^2 f(x_0)}{\partial x_1^2} \cdots \frac{\partial^2 f(x_0)}{\partial x_1 \partial x_N} \\ \\ \frac{\partial^2 f(x_0)}{\partial x_N \partial x_1} \cdots \frac{\partial^2 f(x_0)}{\partial x_N^2} \end{bmatrix}$$

$f(x) = q(x)$

$$\frac{d}{dx}q(x)$$

$$\frac{d}{dx}\left[f(x_0) + g_0^T(x - x_0) + \frac{1}{2}(x - x_0)^T H_0\,(x - x_0)\right]$$

$$g_0 + H_0 \left(x - x_0 \right) = 0$$

$$x^* = x = x_0 - H(x_0)^{-1} g(x_0)$$

$f''(x^*)$

$f(x^*)$

$$H^* = H^* = H_f(x)|_{x=x^*}$$

$H^* > 0$

$H^* < 0$

$f(x$

H^*

$f(x) \neq q(x)$

$$x_{n+1} = x_n + \Delta x_n = x_n - H_n^{-1} g_n = x_n + d_n, \qquad n = 0,\ 1,\ 2, \cdots$$

$g_n = g(x_n)$

$$H_n = H(x_n)$$

$$d_n = \Delta x_n = -H_n^{-1} g_n$$

$g_n = f'(x_n)$

$$H_n = f''(x_n)$$

$$x_{n+1} = x_n - H_n^{-1}\, g_n$$

$f'(x) = g_f(x) = 0$

$O(N^3)$

H_n^{-1}

M

$N < M$

$$\varepsilon(x) = \frac{1}{2}||f(x)||^2 = \frac{1}{2}\sum_{m=1}^{M} f_m^2(x)$$

$$g_\varepsilon(x) = \frac{d}{dx}\varepsilon(x) = \frac{1}{2}\frac{d}{dx}||f(x)||^2 = \frac{df}{dx}\ f = J_f^T\ f$$

$$g_i = \frac{\partial}{\partial x_i}\left(\frac{1}{2}||f||^2\right) = \frac{1}{2}\sum_{m=1}^{M}\frac{\partial}{\partial x_i} f_m^2 = \sum_{m=1}^{M}\frac{\partial f_m}{\partial x_i}\ f_m = \sum_{m=1}^{M} J_{mi}\ f_m$$

$J_{mi} = \partial f_m / \partial x_i$

$J_f(x)$

H_ε

H_{ij}

$$\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{2}||f||^2\right) = \frac{\partial}{\partial x_j} g_i = \sum_{m=1}^{M} \frac{\partial}{\partial x_j}\left[f_m \frac{\partial f_m}{\partial x_i}\right]$$

$$\sum_{m=1}^{M}\left[\frac{\partial f_m}{\partial x_i}\;\frac{\partial f_m}{\partial x_j}+f_m\frac{\partial^2 f_m}{\partial x_i\partial x_j}\right]$$

$$\sum_{m=1}^{M} \frac{\partial f_m}{\partial x_i} \frac{\partial f_m}{\partial x_j} = \sum_{m=1}^{M} J_{mi} J_{mj}$$

J_{ij}

J_f

$f_m \; (\partial^2 f_m / \partial x_i \partial x_j)$

$$H_\varepsilon(x) = \frac{d^2}{dx^2}\varepsilon(x) = \frac{d}{dx}g_\varepsilon(x) \approx J_f^T J_f$$

$\varepsilon(x)$

$$x_{n+1} = x_n - H_n^{-1} g_n \approx x_n - (J_n^T J_n)^{-1} J_n^T f_n$$

$$q(x,y) = \frac{1}{2}[x_1,\ x_2]\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] = \frac{1}{2}(ax_1^2 + bx_1x_2 + cx_2^2)$$

$$g = \begin{bmatrix} ax_1 + bx_2/2 \\ bx_1/2 + cx_2 \end{bmatrix}, \qquad H = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}, \qquad \det H = ac - b^2/4$$

$a = 1$

$c = 2$

b

$b = -2$

$\lambda_1 = 2.618$

$\lambda_2 = 0.382$

$$\det H = \lambda_1 \lambda_2 = 1$$

$H > 0$

$f(0,\, 0) = 0$

$b = 0$

$\lambda_1 = 2$

$\lambda_2 = 1$

$\det H = \lambda_1 \lambda_2 = 2$

$b = 2$

$b = 4$

$\lambda_1 = 3.562$

$\lambda_2 = -0.562$

$\det H = \lambda_1 \lambda_2 = -2$

H

$\delta > 1$

δ

$\delta < 1$

x_n

g_n

x_{n+1}

$N = 3$

$$\begin{cases} f_1(x_1,\, x_2,\, x_3) = 3x_1 - (x_2x_3)^2 - 3/2 \\ f_2(x_1,\, x_2,\, x_3) = 4x_1^2 - 625\, x_2^2 + 2x_2 - 1 \\ f_3(x_1,\, x_2,\, x_3) = exp(-x_1x_2) + 20x_3 + 9 \end{cases}$$

$(x_1 = 0.5,\ x_2 = 0,\ x_3 = -0.5)$

$$J(x) = f^T(x)f(x)$$

$x_0 = 0$

n	$x = [x_1, x_2, x_3]$			$\|\|f(x)\|\|$
0	0.000000,	0.000000,	0.000000	$1.016120e+01$
1	0.500000,	0.500000,	-0.500000	$1.552502e+02$
2	0.499550,	0.250800,	-0.493801	$3.881300e+01$
3	0.500096,	0.126206,	-0.496852	$9.702208e+00$
4	0.500025,	0.063914,	-0.498405	$2.425198e+00$
5	0.500010,	0.032778,	-0.499181	$6.059054e-01$
6	0.500005,	0.017231,	-0.499570	$1.510777e-01$
7	0.500003,	0.009498,	-0.499763	$3.737330e-02$
8	0.500002,	0.005712,	-0.499857	$8.959365e-03$
9	0.500001,	0.003968,	-0.499901	$1.900145e-03$
10	0.500001,	0.003326,	-0.499917	$2.577603e-04$
11	0.500001,	0.003206,	-0.499920	$8.932714e-06$
12	0.500001,	0.003202,	-0.499920	$1.238536e-08$
13	0.500001,	0.003202,	-0.499920	$2.371437e-14$

$J(x) = ||f(x^*)||^2 \approx 10^{-28}$

$$x^* = \begin{bmatrix} 0.5000008539707297 \\ 0.0032017070323056 \\ -0.4999200212218281 \end{bmatrix}$$

$H_f(x)$

H_f

H_f^{-1}

x

$$x = x_0 - \delta f'(x_0) = x_0 + \Delta x_0, \qquad \Delta x_0 = -\delta f'(x_0), \quad \delta > 0$$

$f'(x_0)$

$$f(x) \approx f(x_0) + f'(x_0)\Delta x_0 = f(x_0) - |f'(x_0)|^2\delta < f(x_0)$$

$$x_{n+1} = x_n - \delta_n \, f'(x_n), \qquad n = 0, 1, 2, \cdots$$

$f'(x^*) = 0$

$g(x) = df(x)/dx$

$x = x_0 + \Delta x$

$\Delta x = -\delta g \ (\delta > 0)$

$$f(x) \approx f(x_0) + g_0^T \Delta x = f(x_0) - \delta g_0^T g_0 = f(x_0) - \delta ||g||^2 < f(x_0)$$

$$x_{n+1} = x_n - \delta_n \, g_n = (x_{n-1} - \delta_{n-1} \, g_{n-1}) - \delta_n \, g_n = \cdots = x_0 - \sum_{i=0}^{n} \delta_n \, g_i$$

$g(x^*) = 0$

$$d_n = -H_n^{-1} g_n$$

H_n

$d_n = -g_n$

g

$$f(x) = x^T A x = [x_1 \; x_2] \left[\begin{smallmatrix} a_{11} a_{12} \\ a_{21} a_{22} \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$$

$A = [\begin{smallmatrix}2 & 1\\1 & 1\end{smallmatrix}]$

$a_{12} = a_{21}$

$A \geq 0$

$$f(x_1, x_2) = \frac{1}{2}[x_1 \ x_2] \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{2}(2x_1^2 + 2x_1x_2 + x_2^2)$$

$f(x_1,\, x_2) = 0$

$x_1 = x_2 = 0$

$x_0 = [1,\ 2]^T$

$g_0 = [4,\ 3]^T$

d

$$g = \begin{bmatrix} 2x_1 + x_2 \\ x_1 + x_2 \end{bmatrix}, \quad H = A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad H^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$d_0 = -H^{-1} g_0 = -[1,\ 2]^T$$

$$x_1 = x_0 - H^{-1} g_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$d_0 = -g_0 = -[4,\ 3]^T$$

$$x_1 = x_0 - \delta g_0 = \begin{bmatrix}1\\2\end{bmatrix} - \delta \begin{bmatrix}4\\3\end{bmatrix} = \begin{bmatrix}1 - \delta 4\\2 - \delta 3\end{bmatrix}$$

x_1

$$x_{n+1} = x_n + \delta_n d_n$$

d_n

$f(x_n)$

$\pi/2$

$$\cos^{-1}\left(\frac{d_n^T g_n}{||d_n||\ ||g_n||}\right) > \frac{\pi}{2} \qquad i.e., \qquad d_n^T g_n < 0$$

$$d_n = -H_n^{-1} g_n$$

$$d_n^T g_n = -g_n^T H_n^{-1} g_n < 0, \qquad d_n^T g_n = -g_n^T g_n = -||g_n||^2 < 0$$

δ_n

$$f(x_{n+1}) = f(x_n + \delta_n d_n)$$

$$\frac{d}{d\delta_n} f(x_{n+1}) = \frac{d}{d\delta_n} f(x_n + \delta_n d_n) = \left(\frac{d\, f(x_{n+1})}{dx} \right)^T \; \frac{d(x_n + \delta_n d_n)}{d\delta_n} = g_{n+1}^T d_n = 0$$

$$g_{n+1} = f'(x_{n+1})$$

$$x_{n+1} = x_n + \delta d_n$$

g_{n+1}

$$f(x_{n+1}) = f(x_n + \delta d_n)$$

$\delta = 0$

$f(\delta)$

$$f(x_{n+1}) = f(x_n + \delta d_n)$$

$$[f(x_n + \delta d_n)]_{\delta=0} + \delta \left[\frac{d}{d\delta} f(x_n + \delta d_n)\right]_{\delta=0} + \frac{\delta^2}{2} \left[\frac{d^2}{d\delta^2} f(x_n + \delta d_n)\right]_{\delta=0}$$

$$[f(x_n + \delta d_n)]_{\delta=0}$$

$f(x_n)$

$$\left[\frac{d}{d\delta} f(x_n + \delta d_n)\right]_{\delta=0}$$

$$\left[g(x_n + \delta d_n)^T d_n\right]_{\delta=0} = g_n^T d_n$$

$$\left[\frac{d^2}{d\delta^2} f(x_n + \delta d_n)\right]_{\delta=0}$$

$$\left[\frac{d}{d\delta}g(x_n+\delta d_n)^T\right]_{\delta=0}\frac{d}{d\delta}(x_n+\delta d_n)$$

$$\left[\frac{d}{dx}g(x)\ \frac{d}{d\delta}(x_n+\delta d_n)\right]_{\delta=0}^{T} d_n$$

$(H_n d_n)^T d_n = d_n^T H_n d_n$

$H_n = H_n^T$

$$f(x_n + \delta d_n) \approx f(x_n) + \delta\, g_n^T\, d_n + \frac{\delta^2}{2}\, d_n^T H_n\, d_n$$

$f(x_n + \delta d_n)$

$$\frac{d}{d\delta} f(x_n + \delta d_n)$$

$$\frac{d}{d\delta}\left(f(x_n)+\delta g_n^T\,d_n+\frac{\delta^2}{2}d_n^T H_n\,d_n\right)$$

$$g_n^T \, d_n + \delta \, d_n^T H_n \, d_n = 0$$

$$\delta_n = -\frac{g_n^T d_n}{d_n^T H_n d_n}$$

$$\delta_n = -\frac{g_n^T d_n}{d_n^T H_n d_n} = \frac{g_n^T (H_n^{-1} g_n)}{(H_n^{-1} g_n)^T H_n (H_n^{-1} g_n)} = \frac{g_n^T (H_n^{-1} g_n)}{(H_n^{-1} g_n)^T g_n} = 1$$

$$x_{n+1} = x_n + \delta_n d_n = x_n - \delta_n H_n^{-1} g_n = x_n - H_n^{-1} g_n$$

$$g_{n+1}^T d_n = -g_{n+1}^T g_n = 0, \qquad g_{n+1} \perp g_n$$

$$d_{n+1} = -g_{n+1}$$

$$\delta_n = -\frac{g_n^T d_n}{d_n^T H_n d_n} = \frac{g_n^T g_n}{g_n^T H_n g_n} = \frac{||g_n||^2}{g_n^T H_n g_n}$$

$O(N^2)$

$f''(\delta)\big|_{\delta=0}$

$\delta = \sigma$

σ

$$\left[\frac{d^2}{d\delta^2} f(x + \delta d)\right]_{\delta=0}$$

$$\left[\frac{d}{d\delta} f'(x + \delta d)\right]_{\delta=0} = \lim_{\sigma \to 0} \frac{f'(x + \sigma d) - f'(x)}{\sigma}$$

$$\frac{g^T(x+\sigma d)\,d - g^T(x)\,d}{\sigma} = \frac{g_\sigma^T d - g^T d}{\sigma}$$

$g = g(x)$

$$g_\sigma = g(x + \sigma d)$$

$d_n^T H_n d_n$

$$\frac{d}{d\delta} f(x_n + \delta d_n) \approx g_n^T \, d_n + \frac{\delta}{\sigma} \left(g_{\sigma n}^T d_n - g_n^T d_n \right) = 0$$

$$\delta_n = -\frac{\sigma\, g_n^T d_n}{g_{\sigma n}^T d_n - g_n^T d_n} = -\frac{\sigma\, g_n^T d_n}{(g_{\sigma n} - g_n)^T d_n}$$

$$\delta_n = -\frac{\sigma\, g_n^T d_n}{(g_{\sigma n} - g_n)^T d_n} = -\frac{\sigma\, g_n^T g_n}{(g_{\sigma n} - g_n)^T g_n} = \frac{\sigma\, ||g_n||^2}{||g_n||^2 - g_{\sigma n}^T g_n}$$

$$x_{n+1} = x_n - \delta_n g_n = x_n + \frac{\sigma \, ||g_n||^2}{g_{\sigma n}^T g_n - ||g_n||^2} \, g_n$$

$\sigma = 10^{-6}$

n	$x = [x_1,\ x_2,\ x_3]$			$\|\|f(x)\|\|$
0	0.0000,	0.0000,	0.0000	$1.032500e+02$
10	0.4246,	-0.0073,	-0.5002	$1.535939e-01$
20	0.5015,	0.0064,	-0.4998	$2.448241e-05$
30	0.5009,	0.0057,	-0.4998	$9.178209e-06$
40	0.5006,	0.0052,	-0.4998	$4.17587e-06$
50	0.5004,	0.0049,	-0.4999	$2.122594e-06$
60	0.5003,	0.0047,	-0.4999	$1.182466e-06$
100	0.5001,	0.0043,	-0.4999	$1.805871e-07$
150	0.5000,	0.0041,	-0.4999	$2.720744e-08$
200	0.5000,	0.0041,	-0.4999	$4.934027e-09$
250	0.5000,	0.0040,	-0.4999	$9.630085e-10$
300	0.5000,	0.0040,	-0.4999	$1.939747e-10$
350	0.5000,	0.0040,	-0.4999	$3.961646e-11$
400	0.5000,	0.0040,	-0.4999	$8.140393e-12$
450	0.5000,	0.0040,	-0.4999	$1.677968e-12$
500	0.5000,	0.0040,	-0.4999	$3.462695e-13$
550	0.5000,	0.0040,	-0.4999	$7.136628e-14$
600	0.5000,	0.0040,	-0.4999	$1.474205e-14$

$J(x) = ||f(x^*)||^2 \approx 10^{-14}$

$J(x) \approx 10^{-28}$

$$x^* = \begin{bmatrix} 0.50000136238161 02 \\ 0.00400027495837189 \\ -0.4999000311539049 \end{bmatrix}$$

$$f(x_{n+1}) = f(x_n + \delta d_n) \leq f(x_n) + c_1 \, \delta d^T g_n$$

$$g_{n+1}^T d_n \geq c_2 \ g_n^T d_n$$

$g_n^T d_n < 0$

$$|g_{n+1}^T d_n| < |c_2 \; g_n^T d_n|$$

c_1

c_2

$0 < c_1 < c_2 < 1$

$$\phi(\delta) = f(x_n + \delta d_n)$$

$L_0(\delta) = a + b\delta$

a

$$a = L_0(0) = \phi(\delta)\big|_{\delta=0} = f(x_n)$$

$$b = \frac{d}{d\delta}\ f(x_n + \delta d_n)\bigg|_{\delta=0} = g_n^T d_n < 0$$

$f(x_n + \delta d_n) < f(x_n)$

$$L_0(\delta) = a + b\delta = f(x_n) + g_n^T d_n \delta$$

$L_1(\delta) = f(x_n)$

$L_0(\delta)$

$L_1(\delta)$

$$L(\delta) = f(x_n) + c_1 \, g_n^T d_n \delta$$

$0 < c_1 < 1$

$0 < c_1 g_n^T d_n < g_n^T d_n$

$\delta > 0$

$$f(x_{n+1}) = f(x_n + \delta d_n) \leq f(x_n) + c_1\, g_n^T d_n \delta < f(x_n)$$

$g_{n+1} \perp g_n$

$-g_n$

$f(\delta_n) = f(x_n - \delta_n g_n)$

$f(\delta_n)$

$$g_{n+1}^T g_n = 0$$

$$x_{n+1} = x_n - \delta_n g_n + \alpha_n(x_n - x_{n-1})$$

α_m

$$f(x_1, x_2) = (a - x_1)^2 + b(x_2 - x_1^2)^2 \qquad a = 1,\ b = 100$$

$f(1,\ 1) = 0$

(1, 1)

$$x_{n+1} = x_n - J(x_n)^{-1} f(x_n) = x_n - J_n^{-1} f_n$$

$J_n = J(x_n)$

$f(x_n)$

$$x_{n+1} = x_n - H(x_n)^{-1} g(x_n) = x_n - H_n^{-1} g_n$$

$J(x)$

$H(x)$

$$f(x) = f(x_{n+1}) + (x - x_{n+1})^T g_{n+1} + \frac{1}{2}(x - x_{n+1})^T H_{n+1}(x - x_{n+1}) + O(||x - x_{n+1}||^3)$$

$$\frac{d}{dx}f(x) = g(x) = g_{n+1} + H_{n+1}(x - x_{n+1}) + O(||x - x_{n+1}||^2)$$

$x = x_n$

$g(x_n) = g_n$

$g_{n+1} - g_n$

$H_{n+1}(x_{n+1} - x_n) + O(||x_{n+1} - x_n||^2)$

$B_{n+1}(x_{n+1} - x_n)$

B_n

$$s_n = x_{n+1} - x_n, \qquad y_n = g_{n+1} - g_n$$

$$B_{n+1}s_n = y_n, \qquad B_{n+1}^{-1}y_n = s_n$$

B_n^{-1}

B_0

$n = 0$

$d_n = -B_n^{-1} g_n$

$$B_{n+1} = B_n + \Delta B_n$$

$$B_{n+1}^{-1} = B_n^{-1} + \Delta B_n^{-1}$$

$n = n + 1$

$B_n = I$

$B_n = H_n$

y_n

s_m

$$B_{n+1} = B_n + uu^T$$

u

uu^T

$$B_{n+1}s_n = B_n s_n + uu^T s_n = y_n, \qquad u(u^T s_n) = y_n - B_n s_n$$

$w = y_n - B_n s_n$

$u = cw$

$uu^T s_n = c^2 ww^T s_n = w$

$c = 1/(w^T s_n)^{1/2}$

$$u = cw = \frac{y_n - B_n s_n}{(w^T s_n)^{1/2}} = \frac{y_n - B_n s_n}{((y_n - B_n s_n)^T s_n)^{1/2}}$$

$$B_{n+1} = B_n + uu^T = B_n + \frac{(y_n - B_n s_n)(y_n - B_n s_n)^T}{(y_n - B_n s_n)^T s_n}$$

B_{n+1}

B_{n+1}^{-1}

$$\left(B_n + uu^T\right)^{-1} = B_n^{-1} - \frac{B_n^{-1}uu^TB_n^{-1}}{1 + u^TB_n^{-1}u}$$

$$B_n^{-1} + \frac{(s_n - B_n^{-1} y_n)(s_n - B_n^{-1} y_n)^T}{(s_n - B_n^{-1} y_n)^T y_n}$$

B_{n+1}^{-1}

$$B_{n+1}^{-1} = B_n^{-1} + uu^T$$

$$B_{n+1}^{-1} y_n = B_n^{-1} y_n + u u^T y_n = s_n,$$

$$w^T y_n = (s_n - B_n^{-1} y_n)^T y_n = s_n^T y_n - y_n^T B_n^{-1} y_n > 0$$

$$B_{n+1} = B_n + \alpha u u^T + \beta v v^T$$

$$B_{n+1}s_n = (B_n + \alpha uu^T + \beta vv^T)s_n = B_n s_n + u(\alpha u^T s_n) + v(\beta v^T s_n) = y_n$$

$$u(\alpha u^T s_n) + v(\beta v^T s_n) = y_n - B_n s_n$$

$$u = y_n, \qquad \alpha = \frac{1}{s_n^T y_n}, \qquad v = B_n s_n, \qquad \beta = -\frac{1}{v^T s_n} = -\frac{1}{s_n^T B_n s_n}$$

$$B_{n+1} = B_n + \alpha u u^T + \beta v v^T$$

$$B_{n+1} = B_n + \alpha uu^T + \beta vv^T = B_n + \frac{y_n y_n^T}{y_n^T s_n} - \frac{B_n s_n s_n^T B_n}{s_n^T B_n s_n}$$

$U = [u_1 \ u_2]$

$V = [v_1 \ v_2]$

$$u_1 = v_1 = \frac{y_n}{(s_n^T y_n)^{1/2}}, \qquad u_2 = -v_2 = \frac{B_n s_n}{(s_n^T B_n s_n)^{1/2}}$$

$$B_{n+1} = B_n + u_1 v_1^T + u_2 v_2^T = (B_n + UV^T)^{-1}$$

$$(B_n + UV^T)^{-1} = B_n^{-1} - B_n^{-1}U(I + V^T B_n^{-1}U)^{-1}V^T B_n^{-1}$$

$B_n^{-1} - B_n^{-1} U C^{-1} V^T B_n^{-1}$

$$B_n^{-1} - B_n^{-1}[u_1\ u_2]C^{-1}(B_n^{-1}[v_1\ v_2])^T$$

$$C = \left[\begin{smallmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{smallmatrix}\right] = I + V^T B_n^{-1} U = I + [v_1 \ v_2]^T B_n^{-1} [u_1 \ u_2]$$

$$c_{11} = 1 + v_1^T B_n^{-1} u_1 = 1 + \frac{y_n^T B_n^{-1} y_n}{s_n^T y_n}$$

$$c_{22} = 1 + v_2 B_n^{-1} u_2 = 1 - \frac{s_n^T B_n B_n^{-1} B_n s_n}{s_n^T B_n s_n} = 0$$

$$c_{12} = v_1^T B_n^{-1} u_2 = \frac{y_n^T B_n^{-1} B_n s_n}{(s_n^T y_n)^{1/2} (s_n^T B_n s_n)^{1/2}} = \frac{(s_n^T y_n)^{1/2}}{(s_n^T B_n s_n)^{1/2}}$$

$c_{21} = v_2^T B_n^{-1} u_1 = -c_{12}$

$$C = \begin{bmatrix} c_{11} & c_{12} \\ -c_{12} & 0 \end{bmatrix}, \qquad C^{-1} = \begin{bmatrix} 0 & -1/c_{12} \\ 1/c_{12} & c_{11}/c_{12}^2 \end{bmatrix}$$

C^{-1}

$$B_n^{-1} - [B_n^{-1}u_1 \; B_n^{-1}u_2] \begin{bmatrix} 0 & -1/c_{12} \\ 1/c_{12} & c_{11}/c_{12}^2 \end{bmatrix} [B_n^{-1}v_1 \; B_n^{-1}v_2]^T$$

$$B_n^{-1} - \frac{1}{c_{12}}[B_n^{-1}u_2v_1^TB_n^{-1} - B_n^{-1}u_1v_2^TB_n^{-1}] - \frac{c_{11}}{c_{12}^2}B_n^{-1}u_2v_2^TB_n^{-1}$$

$$\frac{1}{c_{12}} B_n^{-1} u_2 v_1^T B_n^{-1} = \frac{(s_n^T B_n s_n)^{1/2}}{(s_n^T y_n)^{1/2}} B_n^{-1} \frac{B_n s_n}{(s_n^T B_n s_n)^{1/2}} \frac{y_n^T}{(s_n^T y_n)^{1/2}} B_n^{-1} = \frac{s_n y_n^T B_n^{-1}}{s_n^T y_n}$$

$$-\frac{1}{c_{12}}B_n^{-1}u_1v_2^TB_n^{-1} = \frac{(s_n^TB_ns_n)^{1/2}}{(s_n^Ty_n)^{1/2}}B_n^{-1}\frac{y_n}{(s_n^Ty_n)^{1/2}}\frac{s_n^TB_n}{(s_n^TB_ns_n)^{1/2}}B_n^{-1} = \frac{B_n^{-1}y_ns_n^T}{s_n^Ty_n}$$

$$\frac{c_{11}}{c_{12}^2} B_n^{-1} u_2 v_2^T B_n^{-1}$$

$$-\left(1+\frac{y_n^T B_n^{-1} y_n}{s_n^T y_n}\right)\frac{s_n^T B_n s_n}{s_n^T y_n} B_n^{-1}\frac{B_n s_n}{(s_n^T B_n s_n)^{1/2}}\frac{s_n^T B_n}{(s_n^T B_n s_n)^{1/2}} B_n^{-1}$$

$$- \left(1 + \frac{y_n^T B_n^{-1} y_n}{s_n^T y_n}\right) \frac{s_n s_n^T}{s_n^T y_n}$$

$$B_{n+1}^{-1} = B_n^{-1} - \frac{B_n^{-1} y_n s_n^T + s_n y_n^T B_n^{-1}}{s_n^T y_n} + \left(1 + \frac{y_n^T B_n^{-1} y_n}{s_n^T y_n}\right) \frac{s_n s_n^T}{s_n^T y_n}$$

B_n^{-1}

$$B_{n+1}^{-1} = B_n^{-1} + \alpha uu^T + \beta vv^T$$

α

β

$$B_{n+1}^{-1}y_n = (B_n^{-1} + \alpha uu^T + \beta vv^T)y_n = B_n^{-1}y_n + u(\alpha u^T y_n) + v(\beta v^T y_n) = s_n$$

$$u(\alpha u^T y_n) + v(\beta v^T y_n) = s_n - B_n^{-1} y_n$$

$$u = s_n, \qquad \alpha = \frac{1}{s^T y_n}, \qquad v = B_n^{-1} y_n, \qquad \beta = -\frac{1}{v^T y_n} = -\frac{1}{y_n^T B_n^{-1} y_n}$$

$$B_{n+1}^{-1} = B_n^{-1} + \alpha uu^T + \beta vv^T$$

$$B_{n+1}^{-1} = B_n^{-1} + \alpha uu^T + \beta vv^T = B_n^{-1} + \frac{s_n s_n^T}{s_n^T y_n} - \frac{B_n^{-1} y_n y_n^T B_n^{-1}}{y_n^T B_n^{-1} y_n}$$

$$B_{n+1} = B_n - \frac{B_n s_n y_n^T + y_n s_n^T B_n}{y_n^T s_n} + \left(1 + \frac{s_n^T B_n s_n}{y_n^T s_n}\right) \frac{y_n y_n^T}{y_n^T s_n}$$

$z^T B_n z > 0$

$z \neq 0$

$g_{n+1}^T d_n \geq c_2 \; g_n^T d_n \qquad\qquad (g_{n+1} - c_2 \; g_n)^T \, d_n \geq 0$

$$\delta d_n = x_{n+1} - x_n = s_n$$

$$(g_{n+1} - c_2 g_n)^T s_n = g_{n+1}^T s_n - c_2 g_n^T s_n \geq 0$$

$$y_n^T s_n = (g_{n+1} - g_n)^T s_n = g_{n+1}^T s_n - g_n^T s_n$$

$$y_n^T s_n - (g_{n+1}^T s_n - c_2 g_n^T s_n) = (c_2 - 1) g_n^T s_n > 0$$

$g_n^T s_n < 0$

$c_2 < 1$

$$y_n^T s_n \geq g_{n+1}^T s_n - c_2 g_n^T s_n \geq 0$$

$y_n^T s_n > 0$

$B_n = LL^T$

$a = L^T z$

$b = L^T s$

$$a^T a = z^T B_n z, \quad b^T b = s^T B_n s, \quad a^T b = z^T B_n s$$

$$z^T \left[B_n - \frac{B_n s_n s_n^T B_n}{s_n^T B_n s_n} \right] z = z^T B_n z - \frac{(z^T B_n s_n)^2}{s_n^T B_n s_n} = a^T a - \frac{(a^T b)^2}{b^T b} \geq 0$$

$s_n^T y_n \geq 0$

$$z^T \left(\frac{s_n s_n^T}{s_n^T y_n} \right) z \geq 0$$

$$z^T B_{n+1} z = z^T \left(B_n - \frac{B_n s_n s_n^T B_n}{s_n^T B_n s_n} \right) z + z^T \left(\frac{y_n y_n^T}{s_n^T y_n} \right) z \geq 0$$

B^{-1}

N

$$f(x) = f(x_0 + \delta x) \approx f(x_0) + g_0^T \delta x + \frac{1}{2} \delta x^T H_0 \delta x$$

$$f(x) = \frac{1}{2}x^T Ax - b^T x + c$$

$A = H$

$$g(x) = \frac{d}{dx} f(x) = \frac{d}{dx} \left(\frac{1}{2} x^T A x - b^T x + c \right) = Ax - b$$

$$H(x) = \frac{d^2}{dx^2} f(x) = \frac{d}{dx} g = \frac{d}{dx} (Ax - b) = A$$

$$g(x) = Ax - b = 0$$

$x^* = A^{-1}b$

$$f(x^*) = \frac{1}{2}(A^{-1}b)^T A(A^{-1}b) - b^T(A^{-1}b) + c = -\frac{1}{2}b^T A^{-1}b + c$$

$$g(x^*) = Ax^* - b = 0$$

$Ax = b$

$A = A^T$

$u^T A v = (Av)^T u = v^T A u = 0$

$A = I$

$v^T u = 0$

$\{d_0, \cdots, d_{N-1}\}$

$d_i^T A d_j = 0$

$i \neq j$

$$v_1 = \begin{bmatrix}1\\0\end{bmatrix}, \quad v_2 = \begin{bmatrix}0\\1\end{bmatrix}, \quad A = \begin{bmatrix}3 & 1\\1 & 2\end{bmatrix}$$

$$u_1 = v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad u_2 = v_2 - \frac{u_1^T A v_2}{u_1^T A u_1} u_1 = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

$x = [2,\ 3]^T$

v_1

v_2

$$p_{v_1}(x) = \frac{v_1^T x}{v_1^T v_1} v_1 = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \qquad p_{v_2}(x) = \frac{v_2^T x}{v_2^T v_2} v_2 = 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

u_1

u_2

$$p_{u_1}(x) = \frac{u_1^T A x}{u_1^T A u_1} u_1 = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \qquad p_{u_2}(x) = \frac{u_2^T A x}{u_2^T A u_2} u_2 = 3 \begin{bmatrix} -1/3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$x = \begin{bmatrix}2\\3\end{bmatrix}$

$$2\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix} = 2v_1 + 3v_2 = p_{v_1}(x) + p_{v_2}(x)$$

$$3\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}-1/3\\1\end{bmatrix} = 3u_1 + 3u_2 = p_{u_1}(x) + p_{u_2}(x)$$

A

$\{d_0, \cdots, d_n, \cdots\}$

$g_n^T g_{n+1} = 0$

$d_i^T A d_j = 0 \ (i \neq j)$

$$x_{n+1} = x_n + \delta_n d_n = \cdots = x_0 + \sum_{i=0}^{n} \delta_i d_i$$

$e_n = x_n - x^*$

$$e_{n+1} = e_n + \delta_n d_n = \cdots = e_0 + \sum_{i=0}^{n} \delta_i d_i$$

$g = Ax - b$

$g_n = Ax_n - b = A(x^* + e_n) - b = Ax^* - b + Ae_n = Ae_n$

$$\varepsilon = ||g_n||^2$$

δ_i

$$-\frac{d_i^T g_i}{d_i^T A d_i} = -\frac{d_i^T A e_i}{d_i^T A d_i} = -\frac{d_i^T A \left(e_0 + \sum_{j=0}^{i-1} \delta_j d_n\right)}{d_i^T A d_i}$$

$$-\frac{d_i^T A e_0 + \sum_{j=0}^{n-1} \delta_j d_i^T A d_j}{d_i^T A d_i} = -\frac{d_i^T A e_0}{d_i^T A d_i}$$

d_i

d_j

$d_i^T A d_j = 0$

$$e_{n+1} = e_n + \delta_n d_n = e_n - \left(\frac{d_n^T A e_0}{d_n^T A d_n} \right) d_n$$

$e_0 = x_0 - x^*$

$$e_0 = \sum_{i=0}^{N-1} c_i d_i = \sum_{i=0}^{N-1} p_{d_i}(e_0) = \sum_{i=0}^{N-1} \left(\frac{d_i^T A e_0}{d_i^T A d_i} \right) d_i$$

$p_{d_i}(e_0)$

e_0

$$p_{d_i}(e_0) = c_i d_i = \left(\frac{d_i^T A e_0}{d_i^T A d_i} \right) d_i$$

c_i

$$c_i = \frac{d_i^T A e_0}{d_i^T A d_i} = -\delta_i$$

e_{n+1}

$$e_{n+1} = e_0 + \sum_{i=0}^{n} \delta_i d_i = \sum_{i=0}^{N-1} c_i d_i - \sum_{i=0}^{n} c_i d_i = \sum_{i=n+1}^{N-1} c_i d_i = \sum_{i=n+1}^{N-1} p_{d_i}(e_0)$$

$p_{d_n}(e_0)$

$e_N = 0$

$$x_N = x^* + e_N = x^*$$

$d_k^T A$ $(k \leq n)$

$$d_k^T A e_{n+1} = \sum_{i=n+1}^{N-1} c_i d_k^T A d_j = 0$$

$n + 1$

$d_0, \cdots, d_n$

$$d_k^T A e_{n+1} = d_k^T g_{n+1} = 0$$

$N = 2$

$-g_0$

d_1

d_0

e_1

$-g_1$

$\{v_0, \cdots, v_{N-1}\}$

$$d_n = v_n - \sum_{j=0}^{n-1} p_{d_j}(v_n) = v_n - \sum_{m=0}^{n-1} \left(\frac{d_m^T A v_n}{d_m^T A d_m} \right) d_m = v_n - \sum_{m=0}^{n-1} \beta_{nm} d_m$$

$$\beta_{nm} = d_m^T A v_n / (d_m^T A d_m)$$

$\beta_{nm}d_m$

v_n

d_m

$v_n = -g_n$

$$d_n = -g_n - \sum_{m=0}^{n-1} \beta_{nm} d_m,$$

$$\beta_{nm} = \frac{d_m^T A v_n}{d_m^T A d_m} = -\frac{d_m^T A g_n}{d_m^T A d_m} \qquad (m < n)$$

$$g_n = \sum_{i=0}^{n} \alpha_i d_i$$

g_{n+1}^T

$$g_{n+1}^T g_k = g_{n+1}^T \left(\sum_{i=0}^{k} \alpha_i d_i \right) = \sum_{i=0}^{k} \alpha_i \; g_{n+1}^T d_i = 0$$

$g_{n+1}^T d_k = 0$

$g_0, \cdots, g_n$

$g_k^T (k \geq n)$

$$g_k^T d_n = -g_k^T g_n - \sum_{m=0}^{n-1} \beta_{mn}\, g_k^T d_m = -g_k^T g_n = \begin{cases} -||g_n||^2 & n = k \\ 0 & n < k \end{cases}$$

$g_k^T d_m = 0$

$m < n \leq k$

$$g_n^T d_n = -||g_n||^2$$

$$\delta_n = -\frac{g_n^T d_n}{d_n^T A d_n} = \frac{||g_n||^2}{d_n^T A d_n}$$

$$g_{m+1} = Ax_{m+1} - b = A(x_m + \delta_m d_m) - b = (Ax_m - b) + \delta_m A d_m = g_m + \delta_m A d_m$$

g_n^T

$n > m$

$$g_n^T g_{m+1} = g_n^T g_m + \delta_m g_n^T A d_m = \delta_m g_n^T A d_m$$

$g_n^T g_m = 0$

$m \neq n$

$g_n^T A d_m$

$$g_n^T A d_m = \frac{1}{\delta_m} g_n^T g_{m+1} = \begin{cases} ||g_n||^2 / \delta_{n-1} & m = n-1 \\ 0 & m < n-1 \end{cases}$$

$$\beta_{nm} = -\frac{d_m^T A g_n}{d_m^T A d_m} = \begin{cases} -||g_n||^2/\delta_{n-1} d_{n-1}^T A d_{n-1} & m = n-1 \\ 0 & m < n-1 \end{cases}$$

$m = n - 1$

m

β_{nm}

$$d_n = v_n - \sum_{m=0}^{n-1} \beta_{nm} d_m = -g_n - \beta_n d_{n-1}$$

$\delta_{n-1} = ||g_{n-1}||^2 / d_{n-1}^T A d_{n-1}$

$\beta_{nm} = \beta_m$

$$\beta_n = -\frac{||g_n||^2}{||g_{n-1}||^2}$$

$d_0 = -g_0$

$$\delta_n = \frac{||g_n||^2}{d_n^T A d_n}, \qquad x_{n+1} = x_n + \delta_n d_n$$

$$g_{n+1} = \frac{d}{dx} f(x_{n+1})$$

$$\beta_{n+1} = -\frac{||g_{n+1}||^2}{||g_n||^2}$$

$$d_{n+1} = -g_{n+1} - \beta_{n+1} d_n$$

$$f(x,y) = x^T A x = [x_1,\ x_2] \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$x_0 = [1.5,\ -0.75]^T$

n	$x = [x_1,\, x_2]$	$f(x)$
0	1.500000, −0.750000	2.812500
1	0.250000, −0.750000	$0.468750e-01$
2	0.250000, −0.125000	$7.812500e-02$
3	0.041667, −0.125000	$1.302083e-02$
4	0.041667, −0.020833	$2.170139e-03$
5	0.006944, −0.020833	$3.616898e-04$
6	0.006944, −0.003472	$6.028164e-05$
7	0.001157, −0.003472	$1.004694e-05$
8	0.001157, −0.000579	$1.674490e-06$
9	0.000193, −0.000579	$2.790816e-07$
10	0.000193, −0.000096	$4.651361e-08$
11	0.000032, −0.000096	$7.752268e-09$
12	0.000032, −0.000016	$1.292045e-09$
13	0.000005, −0.000016	$2.153408e-10$

n	$x = [x_1,\, x_2]$	$f(x)$
0	1.500000, −0.750000	$2.812500e+00$
1	0.250000, −0.750000	$4.687500e-01$
2	0.000000, −0.000000	$1.155558e-33$

$f(x) = x^T A x$

$$A = \begin{bmatrix} 5 & 3 & 1 \\ 3 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$x_0 = [1,\ 2,\ 3]^T$

$$x_{41} = [3.5486e-06,\ -7.4471e-06,\ 4.6180e-06]^T$$

$f(x) = 8.5429e - 11$

n	$x = [x_1,\ x_2,\ x_3]$	$f(x)$
0	$1.000000, 2.000000, 3.000000$	$4.500000e+01$
1	$-0.734716, -0.106441, 1.265284$	$2.809225e+00$
2	$0.123437, -0.209498, 0.136074$	$3.584736e-02$
3	$-0.000000, 0.000000, 0.000000$	$3.949119e-31$

$N = 9$

$||e|| \approx 10^{-10}$

$||e|| \approx 10^{-16}$

$x_1 = 0.5,\ x_2 = 0,\ x_3 = -0.5$

n	$x = [x_1, x_2, x_3]$	$J(x)$
0	$0.0000, 0.0000, 0.0000$	$1.032500e+02$
1	$0.0113, 0.0050, -0.5001$	$3.160163e+00$
2	$0.0188, -0.0021, -0.5004$	$3.095894e+00$
3	$0.5009, -0.0018, -0.5004$	$7.268252e-05$
4	$0.5009, -0.0017, -0.5000$	$1.051537e-05$
5	$0.5008, -0.0012, -0.5000$	$6.511151e-06$
6	$0.5001, -0.0005, -0.5000$	$6.365321e-07$
7	$0.5001, -0.0005, -0.5000$	$5.667357e-07$
8	$0.5002, -0.0004, -0.5000$	$2.675128e-07$
9	$0.5001, -0.0003, -0.5000$	$1.344218e-07$
10	$0.5001, -0.0002, -0.5000$	$1.241196e-07$
11	$0.5000, -0.0001, -0.5000$	$2.120969e-08$
12	$0.5000, -0.0001, -0.5000$	$1.541814e-08$
13	$0.5000, -0.0001, -0.5000$	$7.282025e-09$
14	$0.5000, -0.0001, -0.5000$	$4.801781e-09$
15	$0.5000, -0.0000, -0.5000$	$4.463926e-09$

$f(x) = x^T Ax/2 - b^T x + c$

$d\,f(x)/dx = Ax - b = 0$

$Ax - b = 0$

$Ax = b$

$d_i,\ (i = 1, \cdots, N)$

$d_i^T A d_j = 0 \ (i \neq j)$

$$x = \sum_{i=1}^{N} c_i d_i$$

$$b = Ax = A\left[\sum_{i=1}^{N} c_i d_i\right] = \sum_{i=1}^{N} c_i A d_i$$

d_j^T

$$d_j^T b = \sum_{i=1}^{N} c_i d_j^T A d_i = c_j d_j^T A d_j$$

c_j

$$c_j = \frac{d_j^T b}{d_j^T A d_j}$$

$$x = \sum_{i=1}^{N} c_i d_i = \sum_{i=1}^{N} \left(\frac{d_i^T b}{d_i^T A d_i} \right) d_i$$

$b = Ax$

$$p_{d_i}(x) = \left(\frac{d_i^T A x}{d_i^T A d_i} \right) d_i$$

$R = N < M$

$A^T A x = A^T b$

$A^T A$

$\Delta E = f(x_{n+1}) - f(x_n)$

$P(\Delta E, T)$

P

$P(\Delta E, T) = e^{-\Delta E/T} >$

T

ΔE

$$
\begin{cases}
f(x) = f(x_1, \cdots, x_N) \\
\begin{cases}
h_i(x) = 0, & (i = 1, \cdots m) \\
g_j(x) \leq 0, & (j = 1, \cdots n)
\end{cases}
\end{cases}
$$

$m + n$

$g_i(x)$

$h_j(x)$

$$\begin{cases} f(x) = f(x_1, \cdots, x_N) \\ h_i(x) = h_i(x_1, \cdots, x_N) = 0, \qquad (i = 1, \cdots m) \end{cases}$$

$m = 1$

$h(x)$

x_1

x_2

$h(x) = 0$

$h(x^*) = 0$

$f(x) = d$

d

$$\bigtriangledown_x f(x^*) = \lambda \bigtriangledown_x h(x^*)$$

$$\bigtriangledown_x f(x) = [\partial f / \partial x_1, \cdots, \partial f / \partial x_N]^T$$

λ

$$L(x, \lambda) = f(x) - \lambda\, h(x)$$

$$\bigtriangledown_{x,\lambda} L(x,\lambda) = \bigtriangledown_{x,\lambda}[f(x) - \lambda\, h(x)] = 0$$

$$\begin{cases} \bigtriangledown_x f(x) = \lambda \bigtriangledown_x h(x) \\ \bigtriangledown_\lambda L(x,\lambda) = \partial L(x,\lambda)/\partial\lambda = -h(x) = 0 \end{cases}$$

$N + m = 2 + 1 = 3$

x_1, x_2, λ

$$\frac{\partial f(x)}{\partial x_i} = \lambda \frac{\partial h(x)}{\partial x_i} \quad (i = 1, 2), \qquad h(x) = 0$$

$x^* = [x_1^*,\, x_2^*]^T$

λ^*

$$\bigtriangledown_x f(x) = \lambda \bigtriangledown_x h(x)$$

$\lambda^* = 0$

$$\bigtriangledown_x f(x^*) = \lambda^* \bigtriangledown_x h(x^*) = 0$$

$\lambda^* \neq 0$

$\bigtriangledown_x f(x^*) \neq 0$

$\bigtriangledown_x h(x^*) \neq 0$

$N > 2$

$h_i(x) = 0 \ (i = 1, \cdots, m)$

$$h_1(x^*) = \cdots = h_m(x^*) = 0$$

$f(x^*) = d$

$h_i(x) = 0$

$\bigtriangledown f(x^*)$

$\bigtriangledown h_i(x^*)$

$$\bigtriangledown_x f(x^*) = \sum_{i=1}^{m} \lambda_i^* \bigtriangledown_x h_i(x^*)$$

$$L(x, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i h_i(x)$$

$\lambda = [\lambda_1, \cdots, \lambda_m]^T$

$$\bigtriangledown_{x,\lambda} L(x,\lambda) = \bigtriangledown_{x,\lambda} \left[f(x) - \sum_{i=1}^{m} \lambda_i h_i(x) \right] = 0$$

$$\bigtriangledown_x f(x) = \sum_{i=1}^{m} \lambda_i \bigtriangledown_x h_i(x) \qquad \frac{\partial f(x)}{\partial x_j} = \sum_{i=1}^{m} \lambda_i \frac{\partial h_i(x)}{\partial x_j} \quad (j = 1, \cdots, N),$$

$$\frac{\partial L(x, \lambda)}{\partial \lambda_i} = h_i(x) = 0 \quad (i = 1, \cdots, m)$$

$\bigtriangledown_x f(x^*)$

$\nabla_x h_i(x^*)$

$\lambda_1, \cdots, \lambda_m$

$\lambda_i = 0$

$i = 1, \cdots, m$

$$\bigtriangledown_x f(x^*) = \sum_{i=1}^{m} \lambda_i \bigtriangledown_x h_i(x) = 0$$

$$\begin{cases} f(x) = f(x_1, \cdots, x_N) \\ g_j(x) = g_j(x_1, \cdots, x_N) \leq \quad \geq 0, \qquad (j = 1, \cdots n) \end{cases}$$

$n = 1$

$g(x) > 0$

$x^* = [x_1^*,\, x_2^*]^T$

$g(x_1, x_2) = 0$

$$\bigtriangledown_x f(x^*) = 0, \qquad g(x^*) \neq 0$$

$\mu^* = 0$

$$\bigtriangledown_x f(x^*) = \mu^* \bigtriangledown_x g(x^*) = 0$$

$$\bigtriangledown_x f(x) = 0$$

$$g(x^*) = 0, \qquad \nabla_x f(x^*) \neq 0$$

$g(x)$

$$\triangledown_x f(x^*) = \mu^* \ \triangledown_x \, g(x^*) \qquad \triangledown_x \, f(x^*) - \mu^* \ \triangledown_x \, g(x^*) = 0$$

$\mu > 0$

$\mu < 0$

μ

$g(x) \geq 0$

$g(x) \leq 0$

x_4

	$g(x) \geq 0$	$g(x) \leq 0$
$\max\ f(x)$	(1) $\bigtriangledown f(x_1) = \mu \bigtriangledown g(x_1),\ \mu < 0$	(3) $\bigtriangledown f(x_3) = \mu \bigtriangledown g(x_3),\ \mu > 0$
$\min\ f(x)$	(2) $\bigtriangledown f(x_2) = \mu \bigtriangledown g(x_2),\ \mu > 0$	(4) $\bigtriangledown f(x_4) = \mu \bigtriangledown g(x_4),\ \mu < 0$

μ

$x_1 \max f(x) g(x) \geq 0 \mu < 0$
$x_2 \min f(x) g(x) \geq 0 \mu > 0$
$x_3 \max f(x) g(x) \leq 0 \mu > 0$
$x_4 \min f(x) g(x) \leq 0 \mu < 0$

$$\mu\, g(x) \begin{cases} \leq 0 \\ \geq 0 \end{cases}$$

$g(x^*) \neq 0$

$g(x^*) = 0$

$\mu^* \neq 0$

$\mu^* g(x^*) = 0$

n

$g_i(x) = 0 \ (i = 1, \cdots, n)$

$$L(x, \mu) = f(x) - \sum_{i=1}^{n} \mu_i \, g_i(x)$$

$\mu = [\mu_1, \cdots, \mu_n]^T$

$(\mu^*)^T \, g(x^*) = 0$

$$\bigtriangledown_{x,\mu} L(x,\mu) = \bigtriangledown_{x,\mu} \left[f(x) - \sum_{i=1}^{n} \mu_i \, g_i(x) \right] = 0$$

$$\bigtriangledown_x f(x) = \sum_{i=1}^{n} \mu_i \bigtriangledown_x g_i(x)$$

$$\bigtriangledown_{\mu}\left[\sum_{i=1}^{n}\mu_i\, g_i(x)\right]=0 \qquad \frac{\partial L(x,\mu)}{\partial \mu_i}=g_i(x)=0 \quad (i=1,\cdots,n)$$

$\bigtriangledown g(x)$

$\mu = 0$

$$f(x) = f(x_1, \cdots, x_N)$$
$$\begin{cases} h_i(x) = 0, & (i = 1, \cdots m) \\ g_j(x) \leq 0 \quad g_j(x) \geq 0, & (j = 1, \cdots n) \end{cases}$$

$f(x)$
$$\begin{cases} h(x) = 0 \\ g(x) \leq 0 \quad g(x) \geq 0 \end{cases}$$

$h(x) = [h_1(x), \cdots, h_m(x)]^T$

$$g(x) = [g_1(x), \cdots, g_n(x)]^T$$

$$L(x,\lambda,\mu) = f(x) - \sum_{i=1}^{m} \lambda_i h_i(x) - \sum_{j=1}^{n} \mu_j g_j(x) = f(x) - \lambda^T h(x) - \mu^T g(x)$$

$L(x, \lambda, \mu)$

λ

μ

$$\frac{\partial}{\partial x}L(x,\lambda,\mu) = 0, \qquad \frac{\partial}{\partial \lambda}L(x,\lambda,\mu) = 0, \qquad \frac{\partial}{\partial \mu}L(x,\lambda,\mu) = 0$$

λ_i

μ_j

$f(x_1, x_2) = x_1^2 + x_2^2$

$x_1 + x_2 = 1$

$x_1 + x_2 \leq 1$

$x_1 + x_2 \geq 1$

$$\begin{cases} f(x_1, x_2) = x_1^2 + x_2^2 \\ h(x_1, x_2) = x_1 + x_2 - 1 = 0 \end{cases}$$

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda g(x_1, x_2) = x_1^2 + x_2^2 - \lambda(x_1 + x_2 - 1)$$

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda = 0, \quad \frac{\partial L}{\partial x_2} = 2x_2 - \lambda = 0, \quad \frac{\partial L}{\partial \lambda} = x_1 + x_2 - 1 = 0$$

$\lambda^* = 1,\ x_1^* = x_2^* = 0.5$

$f(0.5,\, 0.5) = 0.5$

$f(x_1, x_2)$

$g(x_1, x_2)$

$x_1^* = x_2^* = 0.5$

$$\triangledown f(0.5, 0.5) = \triangledown h(0.5, 0.5) = [1, 1]^T$$

$$\begin{cases} f(x_1, x_2) = x_1^2 + x_2^2 \\ g(x_1, x_2) = x_1 + x_2 - 1 \geq 0 \end{cases}$$

$$\begin{cases} f(x_1, x_2) = x_1^2 + x_2^2 \\ g(x_1, x_2) = x_1 + x_2 - 1 \leq 0 \end{cases}$$

$\mu^* = 1 > 0$

$$\begin{cases} f(x_1, x_2) = x_1^2 + x_2^2 \\ g(x_1, x_2) = x_1 + x_2 - 1 \geq 0 \end{cases} \qquad \begin{cases} f(x_1, x_2) = x_1^2 + x_2^2 \\ g(x_1, x_2) = x_1 + x_2 - 1 \leq 0 \end{cases}$$

$$\frac{\partial L}{\partial x_1} = 2x_1 = 0, \quad \frac{\partial L}{\partial x_2} = 2x_2 = 0$$

$x_1^* = x_2^* = 0$

$x^* = 0$

$f(x^*) = 0$

$$\begin{cases} f_p(x) \\ h(x) = 0, \qquad g(x) \leq 0 \qquad g(x) \geq 0 \end{cases}$$

$$L(x, \lambda, \mu) = f_p(x) - \lambda^T h(x) - \mu^T g(x)$$

$$\bigtriangledown_x L(x, \lambda, \mu) = 0, \quad \bigtriangledown_\lambda L(x, \lambda, \mu) = 0, \quad \bigtriangledown_\mu L(x, \lambda, \mu) = 0$$

$$p^* = f_p(x^*)$$

$f_p(x)$

$g_i(x)$

$h_i(x)$

$$f_d(\lambda, \mu) = \min_x L(x, \lambda, \mu) = \min_x \left[f_p(x) - \lambda^T h(x) - \mu^T g(x) \right]$$

$f_d(\lambda, \mu)$

$$\begin{cases} f_d(\lambda,\mu) = \min_x L(x,\lambda,\mu) \\ \mu \leq 0 \qquad\quad \mu \geq 0 \end{cases}$$

$\mu^T g(x) \geq 0$

$\{\lambda^*, \mu^*\}$

$$d^* = f_d(\lambda^*, \mu^*)$$

$\{\lambda,\ \mu\}$

$f_p(x) \geq f_d(\lambda, \mu)$

$$p^* = f_p(x) \geq f_d(\lambda, \mu) = d^*$$

$p^* - d^*$

$p^* - d^* \geq 0$

$p^* = d^*$

d^*

$$\bigtriangledown_x L(x^*, \lambda, \mu) = 0$$

$$h_i(x^*) = 0, \quad g_j(x^*) \leq 0 \qquad g_j(x^*) \geq 0, \qquad (i = 1, \cdots, m, \ j = 1, \cdots, n)$$

$$\mu_j^* \leq 0 \qquad \mu_j^* \geq 0, \quad (j = 1, \cdots, n)$$

μ_j^*

$g_j(x^*)$

$g_j(x^*) \neq 0$

$\mu_j^* = 0$

$\mu_j^* \neq 0$

$g_j(x^*) = 0$

$$(\mu^*)^T g(x^*) = 0$$

$f_p(x^*)$

$g_j(x)$

$h(x^*) = 0$

d^*

$$f_d(\lambda^*, \mu^*) = \min_{x} L(x, \lambda^*, \mu^*)$$

$$\min_{x} \left[f_p(x) - (\lambda^*)^T h(x) - (\mu^*)^T g(x) \right]$$

$\leq$

$$f_p(x^*) - (\lambda^*)^T h(x^*) - (\mu^*)^T g(x^*)$$

$f_p(x^*) - (\mu^*)^T g(x^*)$

$f_p(x^*) = p^*$

$d^* = p^*$

$$(\mu^*)^T g(x^*) = 0$$

$h_j x)$

$$f_d(\lambda^*,\, \mu^*) = L(x^*, \lambda^*,\, \mu^*) = f_p(x^*) - (\lambda^*)^T h(x^*) - (\mu^*)^T g(x^*)$$

$\{(x_n, y_n),\ n = 1, \cdots, N\}$

$$\begin{cases} \frac{1}{2}||w||^2 \\ y_n(x_n^T w + b) \geq 1 \quad (n = 1, \cdots, N) \end{cases}$$

$w = \sum_{n=1}^{N} \alpha_n y_n x_n$

w

$||w||^2$

$\alpha = [\alpha_1, \cdots, \alpha_N]^T$

$$\begin{cases} \sum_{n=1}^{N} \alpha_n - \frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} \alpha_n \alpha_m y_n y_m \left(x_n^T x_m \right) \\ \sum_{n=1}^{N} \alpha_n y_n = y^T \alpha = 0, \quad \alpha_n \geq 0 \quad (n = 1, \cdots, N) \end{cases}$$

$$\alpha^T 1 - \frac{1}{2}\alpha^T Q\alpha$$

Q

$N \times N$

$$Q(m,n) = y_m y_n x_m^T x_n$$

$$f(x_1, \cdots, x_N) = \sum_{i=1}^{N} c_i x_i$$

$x_1, \cdots, x_N$

$$f(x_1, \cdots, x_N) = \sum_{i=1}^{N} c_i x_i$$

$$\begin{cases} \sum_{i=1}^{N} a_{1i} x_i \leq b_1 \\ \cdots\cdots\cdots\cdots \\ \sum_{i=1}^{N} a_{Mi} x_i \leq b_M \\ x_1 \geq 0, \cdots, x_N \geq 0 \end{cases}$$

$$c^T x$$
$$\begin{cases} Ax - b \leq 0 \\ x \geq 0 \end{cases}$$

$$x = \begin{bmatrix} x_1 \\ \\ x_N \end{bmatrix} \qquad c = \begin{bmatrix} c_1 \\ \\ c_N \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ \\ b_M \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} \cdots a_{1N} \\ \\ a_{M1} \cdots a_{MN} \end{bmatrix}$$

$c_1, \cdots, c_N$

a_{ij}

$f(x_1, \cdots, x_N)$

$b_1, \cdots, b_M$

$$f(x_1, x_2, x_3) = 2x_1 - x_2 + 3x_3$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 3 \\ 3x_1 - x_2 + 4x_3 = 10 \\ x_2 \geq 0, \;\; x_3 \geq 0 \end{cases}$$

$x_1 = 3 + 2x_2 - x_3$

$2x_1 - x_2 + x_3 = 3x_2 + x_3 + 6$

$5x_2 + x_3 = 1$

$$f(x_2, x_3) = 3x_2 + x_3 + 6$$
$$\begin{cases} 5x_2 + x_3 = 1 \\ x_2 \geq 0, \quad x_3 \geq 0 \end{cases}$$

$$L(x) = c^T x - y^T (Ax - b) = (c - A^T y)^T x + y^T b$$

$y = [y_1, \cdots, y_M]^T$

$Ax - b \leq 0$

$y \geq 0$

$L(x)$

$$f_d(y) = \max_x L(x) = \max_x [c^T x - y^T (Ax - b)] = \max_x [(c - A^T y)^T x + y^T b]$$

$$\bigtriangledown_x L(x) = \bigtriangledown_x [(c - A^T y)^T x + y^T b] = c - A^T y = 0$$

$L_d(y) = \max_x L(x)$

$f_d(y) = y^T b$

$c - A^T y \leq 0$

$(c - A^T y)^T x \leq 0$

$x \geq 0$

$$f_d(y) = b^T y \geq L(x) = c^T x - y^T(Ax - b) \geq c^T x = f_p(x)$$

y

$$\begin{cases} :c^T x \\ Ax - b \leq 0, \quad x \geq 0 \end{cases} \Longleftrightarrow \begin{cases} b^T y \\ A^T y - c \geq 0, \quad y \geq 0 \end{cases}$$

$$\left\{ \begin{array}{l} :c^T x \\ Ax - b \leq 0, \quad x \geq 0 \end{array} \right.$$

$\Longleftrightarrow$

$$\begin{cases} b^T y \\ A^T y - c \leq 0, \quad y \leq 0 \end{cases}$$

$$\begin{cases} -b^T y \\ A^T y + c \geq 0, \quad y \geq 0 \end{cases}$$

p^*

$$\sum_{i=1}^{N} a_i x_i \leq b \quad \Longrightarrow \quad \sum_{i=1}^{N} a_i x_i + s = b, \quad (s \geq 0)$$

$$z = f(x) = c^T x = \sum_{j=1}^{N} c_j x_j$$

$$\begin{cases} \sum_{i=1}^{N} a_{1i}x_i & +s_1 & & & & = b_1 \\ \sum_{i=1}^{N} a_{2i}x_i & \cdots & +s_2 & & & = b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{i=1}^{N} a_{Mi}x_i & & & & +s_M & = b_M \\ x_1 \geq 0, & \cdots, & x_N \geq 0, & s_1 \geq 0, & \cdots & s_M \geq 0 \end{cases}$$

$N + M$

$\{s_1, \cdots, s_M\}$

$$x = [x_1,\ x_2, \cdots, x_N,\ s_1,\ s_2, \cdots, s_M]^T$$

$M \times (N + M)$

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1N} & 1 & 0 & \cdots 0 \\ a_{21} & a_{22} \cdots a_{1N} & 0 & 1 & \cdots 0 \\ \\ a_{M1} & a_{M2} \cdots a_{MN} & 0 \cdots & 0 & 1 \end{bmatrix}_{M \times (N+M)} = [\ A_{M \times N} \mid I_{M \times M}\ ]$$

$A_{M \times N}$

$I_{M\times M} = [e_1, \cdots, e_M]$

$$c^T x \quad \begin{cases} Ax + s = b \\ x \geq 0,\ s \geq 0 \end{cases} \qquad c^T x \quad \begin{cases} Ax = b \\ x \geq 0 \end{cases}$$

c

$||c|| = 1$

$f(x) = c^T x$

$a_j = [a_{i1}, \cdots, a_{IN}]^T$

$$\sum_{j=1}^{N} a_{ij} x_j = a_j^T x = b_i, \qquad (i = 1, \cdots, M)$$

$x_j \geq 0$

$x \geq 0$

e_j

$$C_{M+N}^{N} = C_{M+N}^{M} = \frac{(M+N)!}{N!\, M!}$$

C_{M+N}^{N}

$x^* \in P$

$\epsilon > 0$

ϵ

$x' = x^* + \epsilon\, c/||c|| \in P$

$f(x')$

$c^T x' = c^T (x^* + \epsilon \, c/||c||)$

$$c^T x^* + \epsilon \, c^T c/||c|| = c^T x^* + \epsilon \, ||c|| > c^T x^* = f(x^*)$$

C_{N+M}^{N}

$$f_p(x_1, x_2) = 2x_1 + 3x_2$$
$$\begin{cases} 2x_1 + x_2 \leq 18 \\ 6x_1 + 5x_2 \leq 60 \\ 2x_1 + 5x_2 \leq 40 \\ x_1 \geq 0, \quad x_2 \geq 0 \end{cases}$$

$$f_p(x) = c^T x$$
$$\begin{cases} Ax - b \leq 0 \\ x \geq 0 \end{cases}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad c = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad b = \begin{bmatrix} 18 \\ 60 \\ 40 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 1 \\ 6 & 5 \\ 2 & 5 \end{bmatrix}$$

$M = 3$

$n + m = 5$

$C_{N+M}^{N} = C_5^2 = 10$

$$f_p(x_1, x_2) = c^T x = c_1 x_1 + c_2 x_2$$

$x = [x_1,\ x_2]^T$

$c = [2,\ 3]^T$

$c^T x = z$

$$
\begin{aligned}
&f_p(x_1, x_2) = 2x_1 + 3x_2 \\
&\begin{cases} 2x_1 + x_2 \leq 18 \\ 6x_1 + 5x_2 \leq 60 \\ 2x_1 + 5x_2 \leq 40 \\ x_1 \geq 0, \;\; x_2 \geq 0 \end{cases}
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
&f_p(x_1, x_2) = 2x_1 + 3x_2 \\
&\begin{cases} 2x_1 + x_2 + s_1 = 18 \\ 6x_1 + 5x_2 + s_2 = 60 \\ 2x_1 + 5x_2 + s_3 = 40 \\ x_1 \geq 0, \; x_2 \geq 0, \; s_1 \geq 0, \; s_2 \geq 0, \; s_3 \geq 0 \end{cases}
\end{aligned}
$$

$$C_{M+N}^{N} = C_5^2 = 10$$

s_1, s_2, s_3

$M + N = 5$

$x^* = (5,\ 6)$

$$f_p(x_1, x_2) = 2x_1 + 3x_2 = 2 \times 5 + 3 \times 6 = 28$$

	x_1	x_2	s_1	s_2	s_3	$c^T x/\|\|c\|\| = 1$
1	5	6	2	0	0	28/7.77
2	6.25	5.5	0	-5	0	29/8.04
3	7.5	3	0	0	10	24/6.66
4	20	0	-2	-60	20	40/11.1
5	10	0	-2	0	20	20/5.55
6	9	0	0	6	22	18/4.10
7	0	8	10	20	0	24/6.66
8	0	12	6	0	-20	36/9.98
9	0	18	0	-30	-50	54/14.99
10	0	0	18	60	40	0/0.00

$$f_d(y_1, y_2, y_3) = 18y_1 + 60y_2 + 40y_3$$
$$\begin{cases} 2y_1 + 6y_2 + 2y_3 \geq 2 \\ y_1 + 5y_2 + 5y_3 \geq 3 \\ y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0 \end{cases}$$

$$f_d(y) = b^T y$$
$$\begin{cases} A^T y - c \geq 0 \\ y \geq 0 \end{cases}$$

$m = 2$

$n = 3$

$y_1 \leq 0$

$y_2 \leq 0$

$y_3 \leq 0$

	(y_1, y_2, y_3)	
1	$(1, 0, 0)$	18
2	$(3, 0, 0)$	54
3	$(0, 1/3, 0)$	20
4	$(0, 3/5, 0)$	36
5	$(0, 0, 1)$	40
6	$(0, 0, 3/5)$	24
7	$(0, 1/5, 2/5)$	28
8	$(1/2, 0, 1/2)$	29
9	$(-2, 1, 0)$	24
10	$(0, 0, 0)$	0

$(0, 1/5, 2/5)$

$$f_d(y_1, y_2, y_3) = 18y_1 + 60y_2 + 40y_3$$

$f_p(x_1, x_2) = 2x_1 + 3x_2$

$(5, 6)$

$A_{M\times N}x_{N\times 1} \leq b_{M\times 1}$

$$Ax = [A_{M \times N} \mid I_{M \times M}]x = b$$

$(A) = M$

I

$$Ax = [A_n \;\; I] \begin{bmatrix} x_n \\ x_b \end{bmatrix} = A_n x_n + I \, x_b = b$$

x_b

A_n

$x_n = 0$

$x_b = b$

$$x = \begin{bmatrix} x_n \\ x_b \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}$$

$Ax \leq b$

$\{e_1, \cdots, e_M\}$

$N + M + 1$

$M + 1$

$\sum_{j=1}^{N} a_{ij} x_j + s_i = b_i$

$i = 1, \cdots, M$

$(M+1)st$

$f(x) = \sum_{j=1}^{N} c_j x_j$

$x_1 = \cdots = x_N = 0$

$$z - c_1 x_1 - \cdots - c_N x_N = 0$$

$(N + M + 1)st$

$b = [b_1, \cdots, b_M]^T$

$z = f(x)$

	x_1	x_2	$\cdots$	x_N	s_1	s_2	$\cdots$	s_M	
s_1	a_{11}	a_{12}	$\cdots$	a_{1N}	1	0	$\cdots$	0	b_1
s_2	a_{21}	a_{22}	$\cdots$	a_{2N}	0	1	$\cdots$	0	b_2
s_M	a_{M1}	a_{M2}	$\cdots$	a_{MN}	0	0	$\cdots$	1	b_M
z	$-c_1$	$-c_2$	$\cdots$	$-c_M$	0	0	$\cdots$	0	0

$x_n = [x_1, \cdots, x_N]^T$

$x_b = [s_1, \cdots, s_M]^T$

$M \times M$

$$x_b = [s_1, \cdots, s_M]^T = b, \qquad x_n = [x_1, \cdots, x_N]^T = 0$$

$$Ax = A_n x_n + I x_b = b$$

x_j

$-c_j$

$c_j = \max\{c_1, \cdots, c_N\}$

$z = \sum_{j=1}^{N} c_j x_j$

$x_k \ \ (k \neq j)$

$\sum_{j=1}^{N} a_{kj} x_j \leq b_k$

$x_j \leq b_k/a_{kj}$

$a_{kj} > 0$

$$b_i/a_{ij} = \min\{b_k/a_{1j}, \cdots, b_k/a_{Mj}\}$$

s_i

$a_{kj} < 0$

$k = 1, \cdots, M$

e_i

r_i

$r_i \leftarrow r_i / a_{ij}$

$a_{ij} = 1$

$a_{kj}r_i$

$r_k \leftarrow r_k - a_{kj} r_i$

$a_{kj} = 0$

$k = 1, \cdots, M + 1$

$k \neq i$

r_{M+1}

b_i

$s_i = 0$

b

$M > N$

$M < N$

$$f(x) = 2x_1 + 3x_2$$

$$\begin{cases} 2x_1 + x_2 + s_1 = 18 \\ 6x_1 + 5x_2 + s_2 = 60 \\ 2x_1 + 5x_2 + s_3 = 40 \\ x_1 \geq 0,\ x_2 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0,\ s_3 \geq 0 \end{cases}$$

y

u

w

b_1

b_2

b_3

	x_1	x_2	s_1	s_2	s_3	b
s_1	2	1	1	0	0	18
s_2	6	5	0	1	0	60
s_3	2	5	0	0	1	40
z	-2	-3	0	0	0	0

$s_1 = 18$

$s_2 = 60$

$s_3 = 40$

$j = 2$

$-c_2 = -3 < -c_1$

r_3

$$b_3/a_{32} = 40/5 = 8 = \min\{18/1 = 18,\ 60/5 = 12,\ 40/5 = 8\}$$

$r_i = r_3 = [2,\ 5,\ 0,\ 0,\ 1,\ 40]$

$a_{ij} = 32 = 5$

$r_i = r_3 = [0.4,\ 1,\ 0,\ 0,\ 0.2,\ 8]$

$a_{kj} r_k = a_{k2} r_k$

r_k

$a_{kj} = a_{k2} = 0, \ (k = 1, \cdots, M + 1 = 4, \ k \neq i)$

r_4

	x_1	x_2	s_1	s_2	s_3	b
s_1	1.6	0	1	0	−0.2	10
s_2	4	0	0	1	−1	20
x_2	0.4	1.0	0	0	0.2	8
z	−0.8	0	0	0	0.6	24

$x_j = x_2$

$s_i = s_3$

$x_1 = 0,\ x_2 = b_3 = 8$

$s_1 = 10$

$s_2 = 20$

$s_3 = 0$

$z = c^T x = 24$

$j = 1$

$$-c_1 = -0.8$$

$i = 2$

$$b_2/a_{21} = 20/4 = 5 = \min\{10/1.6 = 6.25,\ 20/4 = 5,\ 8/0.4 = 20\}$$

$r_i = r_2 = [4,\ 0,\ 0,\ 1,\ -1,\ 20]$

$a_{ij} = a_{21} = 4$

$r_i = r_2 = [1,\ 0,\ 0,\ 0.25,\ -0.25,\ 5]$

$a_{kj}r_k = a_{k1}r_k$

$a_{kj} = a_{k1} = 0, (k = 1, \cdots, M + 1 = 4)$

	x_1	x_2	s_1	s_2	s_3	b
s_1	0	0	1	-0.4	0.2	2
x_1	1	0	0	0.25	-0.25	5
x_2	0	1	0	$-0,1$	0.3	6
z	0	0	0	0.2	0.4	28

s_1

$x_1 = b_2 = 5,\ x_2 = b_3 = 6$

$s_1 = 2, \ s_2 = s_3 = 0$

$z = c^T x = 28$

$z = 0$

$x_1 = 0$

$x_2 = 8$

$z = 24$

$x_1 = 5$

$x_2 = 6$

$z = 28$

$f(x) = c^T x^*$

s

$f(x) = \frac{1}{2}[x - m]^T Q [x - m] = \frac{1}{2} x^T Q x + c^T x + c$
$Ax \leq b$

$Q = Q^T$

$M \times N$

$c = -Qm,\ c = m^T Qm/2$

c

$x^T Q x$

m

$m \leq N$

$Ax - b = 0$

$$L(x, \lambda) = f(x) + \lambda^T (Ax - b) = \frac{1}{2} x^T Q x + c^T x + \lambda^T (Ax - b)$$

$\bigtriangledown_x L(x, \lambda)$

$$Qx + c + A^T\lambda = 0$$

$\bigtriangledown_{\lambda} L(x, \lambda)$

$Ax - b = 0$

$$\begin{bmatrix} Q & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix}$$

$m + N$

λ^*

$f(x_1, x_2) = x_1^2 + x_2^2 = \frac{1}{2}[x_1,\ x_2] \begin{bmatrix} 2 0 \\ 0 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{2} x^T Q x$
$Ax = [1,\ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 + x_2 = b = 1$

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad A = [1,\ 1], \quad c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad b = 1;$$

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$x_1^* = x_2^* = 0.5$

$\lambda^* = -1$

$f(x_1^*, x_2^*) = 0.5$

$M = N$

$$f(x) = \frac{1}{2}x^T Q x + c^T x$$
$$Ax \leq b$$

$f(x)$

$$\begin{cases} h(x) = 0 \\ g(x) \leq 0 \\ x \geq 0 \end{cases} \qquad g(x) \geq 0$$

$s \geq 0$

$$g(x) \leq 0 \implies g(x) + s = 0, \qquad g(x) \geq 0 \implies g(x) - s = 0$$

$h(x) = 0$

$$f(x)$$
$$\begin{cases} h(x) = 0 \\ x \geq 0 \end{cases}$$

$$L(x, \lambda, \mu) = f(x) + \lambda^T h(x) - \mu^T x$$

$$
\begin{cases}
\nabla_x L(x,\lambda,\mu) = g_f(x) + J_h^T(x)\lambda - \mu = 0 \\
h(x) = 0, \quad x \geq 0 \\
\mu \geq 0 \\
\mu_j x_j = 0, \quad (j = 1, \cdots, N) \qquad XM1 = 0
\end{cases}
$$

$$X = \begin{bmatrix} x_1 \cdots & 0 \\ \\ 0 \cdots & x_n \end{bmatrix}_{N \times N}, \qquad M = \begin{bmatrix} \mu_1 \cdots & 0 \\ \\ 0 \cdots & \mu_N \end{bmatrix}_{N \times N}$$

$J_f(x)$

$h(x)$

$$J_h(x) = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} \cdots \frac{\partial h_1}{\partial x_n} \\ \\ \frac{\partial h_m}{\partial x_1} \cdots \frac{\partial h_m}{\partial x_n} \end{bmatrix}_{M \times N}$$

$$I(x) = \begin{cases} 0 & x \geq 0 \\ \infty & x < 0 \end{cases}$$

$x \geq 0$

$f(x) + \sum_{i=1}^{N} I(x_i)$
$h(x) = 0$

$I(x)$

$t > 0$

$$-\frac{1}{t}\ln(x) \quad \stackrel{t\to\infty}{\Longrightarrow} \quad I(x)$$

$f(x) - \frac{1}{t} \sum_{i=1}^{N} \ln x_i$
$h(x) = 0$

$$L(x, \lambda, \mu) = f(x) + \lambda^T h(x) - \frac{1}{t} \sum_{j=1}^{N} \ln x_j$$

$$\bigtriangledown_x L(x, \lambda, \mu) = g_f(x) + J_h^T(x)\lambda - \frac{1}{t}X^{-1}1$$

$$\mu = \frac{x}{t} = \frac{1}{t}X^{-1}1, \qquad X\mu = XM1 = \frac{1}{t}$$

$$\begin{cases} g_f(x) + J_h^T(x)\lambda - \mu = 0 \\ h(x) = 0 \\ XM1 - 1/t = 0 \end{cases}$$

$\mu \geq 0$

$1/t$

$t \to \infty$

$$F(x, \lambda, \mu) = 0$$

J_F

$F(x, \lambda, \mu)$

$$F(x, \lambda, \mu) = \begin{bmatrix} g_f(x) + J_h^T(x)\lambda - \mu \\ h(x) \\ XM1 - 1/t \end{bmatrix}, \qquad J_F = \begin{bmatrix} W(x) & J_h^T(x) & -I \\ J_h(x) & 0 & 0 \\ M & 0 & X \end{bmatrix}$$

$W(x)$

$$\nabla_x \left[g_f(x) + J_h^T(x)\lambda - \mu \right]$$

$$\nabla_x g_f(x) - \nabla_x \left[\sum_{i=1}^{M} \lambda_i \frac{\partial h_i}{\partial x_1}, \cdots, \sum_{i=1}^{M} \lambda_i \frac{\partial h_i}{\partial x_N} \right]^T = H_f(x) - \sum_{i=1}^{M} \lambda_i H_{h_i}(x)$$

$H_{h_i},\ (i = 1, \cdots, M)$

$$\begin{bmatrix} x_{n+1} \\ \lambda_{n+1} \\ \mu_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ \lambda_n \\ \mu_n \end{bmatrix} + \alpha \begin{bmatrix} \delta x_n \\ \delta \lambda_n \\ \delta \mu_n \end{bmatrix}$$

$[\delta x_n, \delta \lambda_n, \delta \mu_n]^T$

$$J_F(x_n, \lambda_n, \mu_n) \begin{bmatrix} \delta x_n \\ \delta \lambda_n \\ \delta \mu_n \end{bmatrix} = \begin{bmatrix} W(x_n) & J_h^T(x_n) & -I \\ J_h(x_n) & 0 & 0 \\ M_n & 0 & X_n \end{bmatrix} \begin{bmatrix} \delta x_n \\ \delta \lambda_n \\ \delta \mu_n \end{bmatrix}$$

$$-F(x_n, \lambda_n, \mu_n) = -\begin{bmatrix} g_f(x_n) + J_h^T(x_n)\lambda_n - \mu_n \\ h(x_n) \\ X_n M_n 1 - 1/t \end{bmatrix}$$

$\mu_0 = x_0/t$

λ_0

X^{-1}

$$M\delta x + X\delta\mu = -XM1 + 1/t$$

$$X^{-1}M\delta x + \delta\mu = -M1 + X^{-1}1/t = -\mu + X^{-1}1/t$$

$$W \delta x + J_h^T(x) \delta \lambda - \delta \mu = -g_f(x) - J_h^T(x) \lambda + \mu$$

$$(W + X^{-1}M)\delta x + J_h^T(x)\delta\lambda = -g_f(x) - J_h^T(x)\lambda + X^{-1}1/t = -\bigtriangledown_x L(x, \lambda, \mu)$$

δx

$\delta\lambda$

$$\begin{bmatrix} W + X^{-1}M & J_h^T(x) \\ J_h(x) & 0 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta \lambda \end{bmatrix} = - \begin{bmatrix} \nabla_x L(x, \lambda, \mu) \\ h(x) \end{bmatrix}$$

$\delta\mu$

$$M\delta x + X\delta\mu = -XM1 - 1/t$$

$$\delta\mu = X^{-1}1/t - X^{-1}M\delta x - \mu$$

$$f(x) = c^T x$$
$$h(x) = Ax - b = 0, \quad x \geq 0$$

$$g_f(x) = \bigtriangledown_x f(x) = c$$

$$J_h(x) = \bigtriangledown_x(Ax - b) = A$$

$$W(x) = \bigtriangledown_x^2 L(x, \lambda, \mu) = 0$$

$$L(x, \lambda, \mu) = c^T x + \lambda^T (Ax - b) - \mu^T x$$

$$
\begin{cases}
\bigtriangledown_x L(x,\lambda,\mu) = g_f(x) + J_h^T(x)\lambda - \mu = c + A^T\lambda - \mu = 0 \\
h(x) = Ax - b = 0 \\
XM1 - 1/t = 0
\end{cases}
$$

$$\begin{bmatrix} 0 & A^T & -I \\ A & 0 & 0 \\ M & 0 & X \end{bmatrix} \begin{bmatrix} \delta x \\ \delta \lambda \\ \delta \mu \end{bmatrix} = - \begin{bmatrix} c + A^T \lambda - \mu \\ Ax - b \\ XM1 - 1/t \end{bmatrix}$$

$$c + A^T\lambda - \mu = 0$$

$$
f(x_1, x_2) = 2x_1 + 3x_2 \quad \begin{cases} 2x_1 + x_2 \le 18 \\ 6x_1 + 5x_2 \le 60 \\ 2x_1 + 5x_2 \le 40 \\ x_1 \ge 0, \;\; x_2 \ge 0 \end{cases} \quad \Longrightarrow \quad f(x_1, x_2, x_3, x_4, x_5) = -2x_1 - 3x_2 \quad \begin{cases} 2x_1 + x_2 + x_3 = 18 \\ 6x_1 + 5x_2 + x_4 = 60 \\ 2x_1 + 5x_2 + x_5 = 40 \\ x_i \ge 0, \;\; (i = 1, \cdots, 5) \end{cases}
$$

$$f(x) = c^T x$$
$$:h(x) = Ax - b = 0, \quad x \geq 0$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad c = \begin{bmatrix} -2 \\ -3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 21100 \\ 65010 \\ 25001 \end{bmatrix}, \quad b = \begin{bmatrix} 18 \\ 60 \\ 40 \end{bmatrix}$$

x_3, x_4, x_5

$x_0 = [1,\ 2,\ 1,\ 1,\ 1]^T$

$t = 9$

$\alpha = 1$

$x^* = [5,\ 6]^T$

$f(x^*) = 2x_1 + 3x_2 = 28$

	$(x_1$	$x_2)$	$f(x)$	
1	$(1.000000e+00$	$2.000000e+00)$	-8.000000	52.413951
2	$(4.654514e+00$	$6.346354e+00)$	-28.348090	3.080204
3	$(5.040828e+00$	$5.946973e+00)$	-27.922575	0.213509
4	$(4.997282e+00$	$6.001764e+00)$	-27.999856	0.004262
5	$(4.999906e+00$	$5.999962e+00)$	-27.999697	0.000303
6	$(4.999989e+00$	$5.999996e+00)$	-27.999966	0.000034
7	$(4.999999e+00$	$6.000000e+00)$	-27.999996	0.000004
8	$(5.000000e+00$	$6.000000e+00)$	-28.000000	0.000000

$x_3 = 2,\ x_4 = x_5 = 0$

$$f(x) = \frac{1}{2}x^T Qx + c^T x$$
$$h(x) = Ax - b = 0, \quad x \geq 0$$

$$g_f(x) = \bigtriangledown_x f(x) = Qx + c$$

$$W(x) = \bigtriangledown_x^2 L(x, \lambda, \mu) = Q$$

$$L(x, \lambda, \mu) = \frac{1}{2}x^T Q x + c^T x + \lambda^T (Ax - b) - \mu^T x$$

$$\begin{bmatrix} Q & A^T & -I \\ A & 0 & 0 \\ M & 0 & X \end{bmatrix} \begin{bmatrix} \delta x \\ \delta \lambda \\ \delta \mu \end{bmatrix} = - \begin{bmatrix} Qx + c + A^T \lambda - \mu \\ Ax - b \\ XM1 - 1/t \end{bmatrix}$$

$$f(x) = [x - m]^T Q [x - m] = x^T Q x + c^T x + d$$
$$h(x) = Ax - b = 0, \quad x \geq 0$$

$$Q = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

$$m = \begin{bmatrix} 4 \\ 10 \end{bmatrix}, \quad \begin{bmatrix} 9 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 7 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

$x_0 = [2,\ 1]^T$

$\delta x, \delta\lambda, \delta\mu$

$\delta x, \delta \lambda$

$$X^{-1}(M\delta x + X\delta\mu) = X^{-1}M\delta x + \delta\mu = -M1 + X^{-1}1/t$$

$$-\mu + [1/x_1 t, \cdots, 1/x_n t]^T = -\mu + \mu = 0$$

$$W(x)\delta x + J_h^T(x)\delta\lambda - \delta\mu = -g_f(x) - J_h^T(x) + \mu$$

$$\begin{bmatrix} W(x) + X^{-1}M & J_h^T(x) \\ J_h(x) & 0 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta \lambda \end{bmatrix} = - \begin{bmatrix} g_f(x) + J_h^T(x)\lambda \\ h(x) \end{bmatrix}$$

$\delta x_n - x_{n+1} - x_n$

$$J_f(x_n)\delta x_n = -f(x_n)$$

$f(x, \lambda, \mu) = 0$

J_f

$(\delta x,\, \delta\lambda,\, \delta\mu)$

$f_0(x)$
$f_j(x) \leq 0, \quad (j = 1, \cdots, n)$ $f(x) \leq 0$

$f(x) = [f_1(x, \cdots, f_n(x)]^T$

$$L(x,\lambda) = f_0(x) - \sum_{j=1}^{n} \lambda_j f_j(x) = f_0(x) - F(x)\lambda$$

$\lambda = [\lambda_1, \cdots, \lambda_n]^T$

$$F(x) = (f_1(x), \cdots, f_n(x))$$

$$\begin{cases} \bigtriangledown f_0(x) + \sum_{j=1}^{n} \lambda_j \bigtriangledown f_j(x) = g_{f_0}(x) + J_f^T(x)\lambda = 0 \\ f(x) \leq 0 \\ \lambda \geq 0 \\ \lambda_j f_j(x) = 0, \quad (j = 1, \cdots, n) \qquad F(x)\lambda = 0 \end{cases}$$

$$f(x) - \frac{1}{t}\sum_{j=1}^{n} \ln(-f_j(x)) = f_0(x) + \frac{1}{t}b(x)$$

$b(x)$

$$b(x) = -\sum_{j=1}^{n} \ln(-f_j(x))$$

$$\bigtriangledown \left[f_0(x) - \frac{1}{t} \sum_{j=1}^{n} \ln(-f_j(x)) \right] = \bigtriangledown f_0(x) - \frac{1}{t} \sum_{j=1}^{n} \frac{1}{f_j(x)} \bigtriangledown f_j(x) = 0$$

$$\lambda_j = -\frac{1}{tf_j(x)} \geq 0, \quad (j = 1, \cdots, n)$$

λ_j

$f_j(x) \leq 0$

$$F(x)\lambda + \frac{1}{t} = 0$$

$$\bigtriangledown f_0(x) + \sum_{j=1}^{n} \lambda_j \bigtriangledown f_j(x) = g_{f_0}(x) + J_f^T(x)\lambda = 0$$

$$L_d(x, \lambda) = f_0(x) + \sum_{j=1}^{n} \lambda_j f_j(x)$$

x^*, λ^*

$L_d(x, \lambda)$

$$\begin{cases} y_1(x,\lambda) = g_{f_0}(x) + J_f^T(x)\lambda = 0 \\ y_2(x,\lambda) = 1/t + F(x)\lambda = 0 \end{cases}$$

$f(x) \leq 0$

$\lambda \geq 0$

$t = t_0$

t

$x^*(t)$

$f_0(x)$

$$\begin{bmatrix}\partial y_1/\partial x \partial y_1/\partial \lambda \\ \partial y_2/\partial x \partial y_2/\partial \lambda\end{bmatrix} \begin{bmatrix}\delta x \\ \delta \lambda\end{bmatrix} = \begin{bmatrix}H_{f_0}(x) J_f^T(x) \\ J_f(x) \;\; F(x)\end{bmatrix} \begin{bmatrix}\delta x \\ \delta \lambda\end{bmatrix} = - \begin{bmatrix}y_1 \\ y_2\end{bmatrix} = - \left[g_{f_0}(x) + J_f^T(x)\lambda 1/t + F(x)\lambda\right]$$

$x + \delta x$

$\lambda + \delta\lambda$

$\lambda^*(t)$

$$L_d(x^*, \lambda^*) = f_0(x^*) + \sum_{j=1}^{n} \lambda_j f_j(x^*) = f_0(x^*) - \sum_{j=1}^{n} \frac{f_j(x^*)}{t\, f_j(x^*)} = f_0(x^*) - \frac{n}{t} < f_0(x^*)$$

$$f_0(x^*) - L_d(x^*, \lambda^*) = p^* - d^* = n/t$$