The Sampling Theory — E186 Handout

Fourier expansion and Fourier transform

- **Fourier expansion**

  Let $x_T(t)$ represent a continuous periodic function

  $$x_T(t+mT) = x_T(t), \qquad (m = 0,\ \pm 1,\ \pm 2, \cdots)$$

  with its period equal to T. Its Fourier expansion is

  $$x_T(t) = \sum_{n=-\infty}^{\infty} X(n) e^{j2\pi n f_0 t}$$

  where $f_0 = 1/T$ and $X(n)$, $(n = 0, \pm 1, \pm 2, \cdots)$ are the expansion coefficients defined as

  $$X(n) = \frac{1}{T} \int_{-T/2}^{+T/2} x_T(t) e^{-j2\pi n f_0 t} dt$$

  These coefficients can also be considered as the discrete, non-periodic spectrum of $x_T(t)$

  $$X(f) = \sum_{n=-\infty}^{\infty} X(n) \delta(f - n f_0)$$

  where $f_0 = 1/T$ is the interval between two neighboring frequency components.

- **Fourier transform of Non-periodic, continuous function**

  The Fourier of a non-periodic, continuous function $x(t)$ is

  $$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

  where $X(f)$ is the non-periodic continuous spectrum

  $$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

  These two equations are a Fourier transform pair (inverse and forward).

- **The convolution theorem** The convolution of two functions $x(t)$ and $y(t)$ is defined as

$$x(t) * y(t) \triangleq \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau = \int_{-\infty}^{\infty} y(\tau)x(t-\tau)d\tau$$

It can be shown that the spectrum of $x(t) * y(t)$ is the product of their spectra $X(f)$ and $Y(f)$:

$$\int_{-\infty}^{\infty} [x(t) * y(t)]e^{-j2\pi ft}dt = X(f)Y(f)$$

Also, the spectrum of the product of the two functions is the convolution of their spectra:

$$\int_{-\infty}^{\infty} [x(t)\, y(t)]e^{-j2\pi ft}dt = X(f) * Y(f)$$

The comb function and its spectrum

The comb (or sampling) function is a sequency of infinite delta impulses uniformly distributed in time (or space):

$$comb(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT)$$

where T is the interval between two neighboring impulses, and $f_0 = 1/T$ is the *sampling rate.*

As $comb(t)$ is periodic (period T), it can be Fourier expanded:

$$comb(t) = \sum_{n=-\infty}^{\infty} COMB(n) e^{j2\pi n f_0 t}$$

where the expansion coefficients are

$$\begin{aligned} COMB(n) &= \frac{1}{T}\int_{-T/2}^{+T/2} x_T(t) e^{-j2\pi n f_0 t} dt \\ &= \frac{1}{T}\int_{-T/2}^{+T/2} [\sum_{m=-\infty}^{\infty} \delta(t - mT)] e^{-j2\pi n f_0 t} dt \\ &= \frac{1}{T}\int_{-T/2}^{+T/2} \delta(t) e^{-j2\pi n f_0 t} dt = \frac{1}{T} e^0 = \frac{1}{T} \qquad (n = 0, \pm 1, \pm 2, \cdots) \end{aligned}$$

Now the expression for $comb(t)$ can be written as:

$$comb(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT) = \sum_{n=-\infty}^{\infty} COMB(n) e^{j2\pi n f_0 t} = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{j2\pi n f_0 t}$$

Again, the coefficients can be considered as the discrete, non-periodic spectrum of the comb function:

$$COMB(f) = \sum_{n=-\infty}^{\infty} COMB(n) \delta(f - n f_0) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta(f - n f_0)$$

Note that the interval between two neighboring impulses is the sampling rate $f_0 = 1/T$. That is, high sampling rate (small interval between two neighboring sampling impulses) of $comb(t)$ corresponds to large gap T in it spectrum.

The sampling theorem

The sampling process of a continuous function $x(t)$ can be represented by

$$x_s(t) = x(t)\, comb(t)$$

According to the convolution theorem, the spectrum of the resultant signal is

$$X_s(f) = X(f) * COMB(f)$$

This convolution is very easy to carry out as $COMB(f)$ is composed of a sequency of impulses. Note that after sampling $x_s(t)$ becomes discrete and its spectrum becomes periodic accordingly.

Now the sampling theorem can be easily obtained by observing that only when the sampling rate f_0 is more than twice the highest frequency component f_{max} of the signal, can the signal be recovered from its discrete samples (by ideal low-pass filtering). Otherwise, some high frequency components in the signal will be mixed with some low frequency components, i.e., *aliasing* occurs, and the signal can never be recovered.

Obviously two things can be done to eliminate or reduce aliasing:

- Increase the sampling rate $f_0 = 1/T$ by reducing T;
- Reduce the high frequency component f_{max} contained in the signal by low-pass filtering it before sampling.