

Restoration of Motion Blurred Images — E186 Handout

Motion Model

The general imaging process can be modeled by

$$g(x, y) = \int_0^T \int \int_{-\infty}^{\infty} h(x, y, x', y', t) f(x', y', t) dx' dy' dt$$

where T is the exposure time. If the imaging system is ideal, spatial and time invariant, i.e.,

$$h(x, y, x', y', t) = \delta(x - x', y - y')$$

then the imaging process becomes

$$g(x, y) = \int_0^T f(x, y, t) dt$$

If the signal is also time invariant, i.e., $f(x, y, t) = f(x, y)$, the image we get is simply

$$g(x, y) = T f(x, y)$$

However, this is not true if there exists some relative motion between the object and the camera system as the signal $f(x, y, t)$ is no longer time invariant.

The planar relative motion can be described by $\{x_d(t), y_d(t)\}$, its two components in x and y directions, respectively, and the image of this moving object is

$$g(x, y) = \int_0^T f(x, y, t) dt = \int_0^T f(x - x_d(t), y - y_d(t)) dt$$

For simplicity, we assume 1D linear motion in x direction only, i.e.,

$$x_d(t) = vt, \quad y_d(t) = 0$$

where v is the speed of the motion.

If we introduce a new variable $x' = vt$ and realize that $dt = dx'/v$ and the integral from 0 to T with respect to t becomes integral from 0 to $L \triangleq vT$ with respect to x' , the imaging process can be described as

$$\begin{aligned} g(x) &= \int_0^T f(x, y, t) dt = \int_0^T f(x - vt) dt \\ &= \frac{1}{v} \int_0^L f(x - x') dx' = \int_{-\infty}^{\infty} f(x - x') h(x') dx' \\ &= f(x) * h(x) \end{aligned}$$

where

$$h(x) \triangleq \begin{cases} 1/v & \text{if } 0 \leq x \leq L \\ 0 & \text{else} \end{cases}$$

can also be considered as the point spread function (PSF) of the imaging system.

Restoration by Inverse Filtering

The above convolution becomes multiplication if we Fourier transform both sides of the equation into frequency domain

$$G(f_x) = F(f_x)H(f_x)$$

where G , F , and H are the spectra of g , f and h , respectively. Specifically, we have

$$H(f_x) = \int_{-\infty}^{\infty} h(x) e^{-j2\pi x f_x} dx = \int_0^L e^{-j2\pi x f_x} dx = e^{-j\pi L f_x} \frac{\sin(\pi L f_x)}{\pi f_x v}$$

While the inverse filtering method could be applied to restore $f(x)$ by inverse transforming $F(f_x)$

$$F(f_x) = \frac{G(f_x)}{H(f_x)}$$

we also realize that those points of $F(f_x)$ corresponding to $H(f_x) = 0$ at $f_x = k/L$, ($k = \pm 1, \pm 2, \dots$) can never be restored. Interpolation from the neighboring points would not work (why?).

Moreover, this inverse filtering method is sensitive to noise that may exist in the imaging process.

Restoration by Spatial Differentiation

To simplify the problem we assume:

- The image is blurred by linear motion:

$$g(x) = \int_0^T f(x - x_d(t))dt = \int_0^T f(x - vt)dt$$

where v is the constant speed of the motion and $L = vT$ is the distance traveled during the exposure time T .

- The width of the image W is a multiple of L :

$$W = KL$$

We next introduce a new variable $x' = x - vt$, and have $t = (x - x')/v$ and $dt = -dx'/v$. Moreover, the integral limits 0 and T for t become, respectively, x and $x - vT = x - L$ for x' . Now the image becomes

$$g(x) = \int_0^T f(x - vt)dt = -\frac{1}{v} \int_x^{x-L} f(x')dx' = \frac{1}{v}[F(x) - F(x - L)]$$

where

$$F(x) = \int f(x')dx'$$

For convenience, we will ignore the constant factor $1/v$.

As the motion distortion is essentially an integration $g(x) = \int f(x')dx'$, to restore $f(x)$ from $g(x)$, we can simply differentiate $g(x)$:

$$g'(x) = \frac{d}{dx}g(x) = f(x) - f(x - L)$$

and restore the original signal $f(x)$ as

$$f(x) = g'(x) + f(x - L) \quad \text{for } 0 \leq x \leq L$$

Note that above equation only recovers $f(x)$ inside the interval $0 \leq x \leq L$.

To recover the rest of $f(x)$, we replace x by $x+mL$ for $m = 0, 1, \dots, K-1$ and apply the above relationship recursively

$$\begin{aligned}
f(x+mL) &= g'(x+mL) + f(x+(m-1)L) \\
&= g'(x+mL) + g'(x+(m-1)L) + f(x+(m-2)L) \\
&= \dots\dots\dots \\
&= g'(x+mL) + g'(x+(m-1)L) + \dots + g'(x) + f(x-L) \\
&= \sum_{i=0}^m g'(x+iL) + f(x-L) \quad (m = 0, 1, \dots, K-1)
\end{aligned}$$

Here $f(x-L)$ represents the segment of signal of length L that moves from outside the image into the image during the exposure time T . If $f(x-L)$ is known, for example, if we can assume $f(x-L) = \text{constant}$ (e.g., uniform background), then the original signal $f(x)$ over the entire interval $0 \leq x \leq KL = W$ can be obtained by evaluating the above equation at $0 \leq x \leq L$ for all $m = 0, 1, \dots, K-1$.

However, if we cannot assume $f(x-L) = \text{constant}$, it need be estimated. As the above equation is valid for $m = 0, 1, \dots, K-1$, we actually have K equations which can be added up to give

$$\sum_{m=0}^{K-1} f(x+mL) = \sum_{m=0}^{K-1} \sum_{i=0}^m g'(x+iL) + \sum_{m=0}^{K-1} f(x-L)$$

which can be solved for $f(x-L)$

$$f(x-L) = \frac{1}{K} \sum_{m=0}^{K-1} f(x+mL) - \frac{1}{K} \sum_{m=0}^{K-1} \sum_{i=0}^m g'(x+iL)$$

The first term on the right is an average of $f(x)$ over the entire range of the image and can be estimated by the average of $g(x)$.

Numerical Derivatives

- **Goal:**

Given a set of $n + 1$ points $\{(x_i, y_i), \quad | \quad (i = 0, \dots, n)\}$, we want to estimate the derivative dy/dx at these points.

- **Lagrange interpolation:** We first reconstruct a continuous function $y = f(x)$ so that

$$f(x_i) = y_i \quad (i = 0, \dots, n)$$

using Lagrange formula:

$$P_n(x) = \sum_{i=0}^n \prod_{(j=0, j \neq i)}^n \frac{(x - x_j)}{(x_i - x_j)} y_i$$

$P_n(x)$ is a polynomial function of x of order n which passes through all given points, i.e., $P_n(x_i) = y_i, \quad (i = 0, \dots, n)$.

For example, when $n = 2$, we have

$$P_3(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

- **Derivative estimation:**

Now the numerical derivative can be found by differentiating $P_n(x)$.

For example,

$$P'_3(x) = \frac{2x - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{2x - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{2x - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)} y_2$$

Specially when the x_i 's are equally spaced, i.e. $x_{i-1} - x_i = h$, the estimated derivative at $x = x_0$ can be found to be

$$p'_3(x_0) = (-3y_0 + 4y_1 - y_2)/2h$$

It can be shown that the error is $h^2 f'''(\xi)/3$. Here $f'(x_0)$ is estimated using 3 points $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ and is therefore called the forward estimation.

We can also get backward estimation of $f'(x_0)$ using $(x_0, y_0), (x_{-1}, y_{-1}), (x_{-2}, y_{-2})$:

$$p'_3(x_0) = (3y_0 - 4y_1 + y_2)/2h$$

The average of the forward and backward estimations can be used as the final estimation of $f'(x_0)$.

To overcome possible noise, better estimation may be obtained by using more points, such as 4 or 5 points.

The four-point forward estimation:

$$p'_4(x_0) = (2y_3 - 9y_2 + 18y_1 - 11y_0)/6h$$

The four-point backward estimation:

$$p'_4(x_0) = (-2y_{-3} + 9y_{-2} - 18y_{-1} + 11y_0)/6h$$

The five-point forward estimation:

$$p'_5(x_0) = (-3y_4 + 16y_3 - 36y_2 + 48y_1 - 25y_0)/12h$$

The five-point backward estimation:

$$p'_5(x_0) = (3y_{-4} - 16y_{-3} + 36y_{-2} - 48y_{-1} + 25y_0)/12h$$

This method is based on the assumption that the function $y = f(x)$ is continuous and noise-free. If this is not the case, the estimation error could be large.