Hough Transform — E186 Handout

Definition

The idea of Hough transform is to describe a certain line shape (straight lines, circles, ellipses, etc.) globally in a parameter space – the Hough transform domain. We assume the shape can be described by an equation

$$f(x, y, \alpha_1, \cdots, \alpha_n) = 0$$

where $\{\alpha_1, \cdots, \alpha_n\}$ are a set of n parameters needed to describe the shape.

- **The forward transform**

  Every point (x, y) in the spatial domain is transformed to a hyper-surface in the n-dimensional parameter space specified by the above equation.

- **The inverse transform**

  Every point $(\alpha_1, \cdots, \alpha_n)$ in the parameter space describes by the same equation a specific curve, the shape of interest, in the 2D image plane.

Straight Line Detection

Any straight line in 2D space can be represented by this parametric equation:

$$f(x, y, \rho, \theta) = x\ cos\ \theta + y\ sin\ \theta - \rho = 0$$

To find the transform of a certain point (x_0, y_0) in the parameter space, we solve the equation above for ρ and get

$$\begin{aligned}
\rho &= x_0\ cos\ \theta + y_0\ sin\ \theta - \rho \\
&= \sqrt{x_0^2 + y_0^2}\ \left(\frac{x_0}{\sqrt{x_0^2 + y_0^2}}\ cos\ \theta + \frac{y_0}{\sqrt{x_0^2 + y_0^2}}\ sin\ \theta\right) \\
&= r_0\ (cos\ \alpha_0\ cos\ \theta + sin\ \alpha_0\ sin\ \theta) \\
&= r_0\ cos(\alpha_0 - \theta)
\end{aligned}$$

where

$$\begin{cases} r_0 \triangleq \sqrt{x_0^2 + y_0^2} \\ \alpha_0 \triangleq tan^{-1}(y_0/x_0) \end{cases}$$

Now we see that the transform of the point (x_0, y_0) is a sinusoidal curve in the parameter space. And a given point (ρ_0, θ_0) in the parameter space can be inverse transformed back to the spatial domain to represent a straight line specified by the equation

$$x\ cos\ \theta_0 + y\ sin\ \theta_0 - \rho_0 = 0$$

To detect all straight lines in the image, the following algorithm can be used.

- Make available an $n = 2$ dimensional array

  $$H(\rho_k, \theta_l), \quad (0 \le k, l \le m)$$

  for the parameter space. Here m represents the resolution of the parameter space.

- Find the gradient image of the given image:

  $$G(x_i, y_j) = |G(x_i, y_j)| \angle G(x_i, y_j), \quad (0 \le i, j \le n)$$

  where n is the resolution of the image.

- For any pixel satisfying $|G(x_i, y_j)| > T_s$, increment all elements on the curve $\rho = x_i \, cos \, \theta + y_i \, sin \, \theta$ in the parameter space:

for all θ

{

$$\rho = x_i \, cos \, \theta + y_j \, sin \, \theta;$$

$$H(\rho, \theta) = H(\rho, \theta) + 1;$$

}

- In the parameter space, any element $H(\rho, \theta) > T_h$ represents a straight line detected in the image.

This algorithm can be improved by making use of the gradient direction $\angle G$, as in this case, the gradient direction is the same as the angle θ. Therefore for any point $|G(x, y)| > T_s$, we only need to increment the elements on a small segment of the sinusoidal curve. The third step in the algorithm can be modified as

- For any pixel satisfying $|G(x_i, y_j)| > T_s$,
  for all θ satisfying $\angle G(x_i, y_j) - \Delta\theta \le \theta \le \angle G(x_i, y_j) + \Delta\theta$
  {

$$\rho = x_i \, cos \, \theta + y_j \, sin \, \theta;$$

$$H(\rho, \theta) = H(\rho, \theta) + 1;$$

  }

where $\Delta\theta$ defines a small range in θ to allow some room for error in $\angle G$.

Make Use of $\angle G$

In general, the gradient direction $\angle G$ can be used to reduce the computation in the parameter space. We first note that the tangent direction ϕ of any function $f(x, y)$ is related to the derivative dy/dx

$$tan\ \phi = \frac{dy}{dx} = -\frac{f'_x(x, y)}{f'_y(x, y)} \triangleq g(x, y)$$

where $f'_x(x, y)$ and $f'_y(x, y)$ are the partial derivatives of the function $f(x, y)$ with respect to x and y, respectively.

We also note that at an arbitrary point on the curve specified by the equation $f(x, y) = 0$, the gradient direction is always perpendicular to the tangent direction:

$$\phi = \angle G \pm \frac{\pi}{2}$$

i.e.

$$tan\ \phi = tan\ (\angle G \pm \frac{\pi}{2}) = -cot\ \angle G$$

Now the gradient direction $\angle G$ can be used to establish a second equation, in addition to the original equation describing the shape, and thereby reducing the dimensions of the hyper-surface in the parameter space that needs to be incremented by 1.

$$\left\{ \begin{array}{l} f(x, y, \alpha_1, \cdots, \alpha_n) = 0 \\ g(x, y, \alpha_1, \cdots, \alpha_n) = -cot\ \angle G \end{array} \right.$$

Detection of Circles

A circle in the image can be described by

$$f(x, y, x_0, y_0, r) = (x - x_0)^2 + (y - y_0)^2 - r^2 = 0$$

where x_0, y_0, and r are three parameters which span a 3D parameter Hough space. Any point (x, y) in the image corresponds to a cone shaped surface in the 3D parameter space.

To use $\angle G$, consider the derivative

$$\frac{dy}{dx} = -\frac{f'_x(x, y)}{f'_y(x, y)} = -\frac{x - x_0}{y - y_0}$$

Now we only need to increment those elements in the parameter space that satisfy both of the following equations

$$\begin{cases} (x - x_0)^2 + (y - y_0)^2 - r^2 = 0 \\ (x - x_0)/(y - y_0) = cot\ \angle G \end{cases}$$

Geometrically these two simultaneous equations represent the intersection of a cone specified by the first equation and a plane specified by the second equation which passes through the axis of the cone.

Solving these equations for x_0 and y_0, we get

$$\begin{cases} x_0 = x \pm r\ cos\ \angle G \\ y_0 = y \pm r\ sin\ \angle G \end{cases}$$

and the algorithm for detecting circles:

- For any pixel satisfying $|G(x_i, y_j)| > T_s$, increment all elements satisfying the two simultaneous equations above;

  for all r

  {

  $$\begin{cases} x_0 = x_i \pm r\ cos\ \angle G \\ y_0 = y_j \pm r\ sin\ \angle G \end{cases}$$

  $$H(x_0, y_0, r) = H(x_0, y_0, r) + 1;$$

  }

- In the parameter space, any element $H(x_0, y_0, r) > T_h$ represents a circle with radius r located at (x_0, y_0) in the image.

Detection of Ellipses

Here we assume the axes of the ellipses are in parallel with the coordinates of the image space, i.e., the equations specifying the ellipses are in the following standard form

$$f(x, y, x_0, y_0, a, b) = \frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} - 1 = 0$$

where x_0, y_0, a and b are four parameters which span a 4D parameter Hough space.

To use $\angle G$, consider

$$\frac{dy}{dx} = -\frac{f'_x(x, y)}{f'_y(x, y)} = -\frac{(x - x_0)/a}{(y - y_0)/b}$$

Now we only need to increment those elements in the parameter space that satisfy both of the following equations

$$\begin{cases} \frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} - 1 = 0 \\ \frac{(x-x_0)/a}{(y-y_0)/b} = cot\ \angle G \end{cases}$$

Solving these equations for x_0 and y_0, we get

$$\begin{cases} x_0 = x_i \pm a\ cos\ \angle G \\ y_0 = y_j \pm b\ sin\ \angle G \end{cases}$$

and the algorithm for detecting circles:

- For any pixel satisfying $|G(x_i, y_j)| > T_s$, increment all elements satisfying the two simultaneous equations above;

  for all a and all b

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  $$\begin{cases} x_0 = x \pm a\ cos\ \angle G \\ y_0 = y \pm b\ sin\ \angle G \end{cases}$$

  $$H(x_0, y_0, a, b) = H(x_0, y_0, a, b) + 1;$$

  }

- In the parameter space, any element $H(x_0, y_0, a, b) > T_h$ represents an ellipse detected in the image.