Fourier Transform — E180 Handout

Four different forms of the Fourier transform

- **Non-periodic, continuous time function $x(t)$, continuous, non-periodic spectrum $X(f)$**

  This is the most general form of Fourier transform.

  $$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt$$

  $$x(t) = \int_{-\infty}^{+\infty} X(f) e^{j2\pi ft} df$$

  The first one is the forward transform, and the second one is the inverse transform.

- **Non-periodic, discrete time function $x(n)$, continuous, periodic spectrum $X_F(f)$**

  The discrete time function can be considered as a sequence of samples of continuous time function. The time interval between two consecutive samples $x(m)$ and $x(m+1)$ is $t_0 = 1/F$, where F is the sampling rate, which is also the period of the spectrum in the frequency domain.

  The discrete time function can be written as

  $$x(t) = \sum_{m=-\infty}^{+\infty} x(m) \delta(t - mt_0)$$

  and its transform is:

  $$X_F(f) = \sum_{m=-\infty}^{+\infty} x(m) e^{-j2\pi fmt_0}$$

  $$x(m) = \frac{1}{F} \int_{-F/2}^{+F/2} X_F(f) e^{j2\pi fmt_0} df$$

  $$(m = 0, \pm 1, \pm 2, \cdots)$$

We can verify that the spectrum is indeed periodic:

$$X_F(f+kF) = X_F(f+k/t_0) = \sum_{m=-\infty}^{+\infty} x(m)e^{-j2\pi(f+k/t_0)mt_0} = X_F(f)$$

(for $k = \pm 1, \pm 2, \cdots$) because $e^{\pm j2\pi mk} = 1$.

- **Periodic, continuous time function $x_T(t)$, discrete, non-periodic spectrum $X(n)$**

  This is the Fourier series expansion of periodic functions. The time period is T, and the interval between two consecutive frequency components is $f_0 = 1/T$, and its transform is:

$$X(n) = \frac{1}{T}\int_{-T/2}^{+T/2} x_T(t)e^{-j2\pi nf_0 t}dt$$

$$x_T(t) = \sum_{n=-\infty}^{+\infty} X(n)e^{j2\pi nf_0 t}$$

$$n = 0, \pm 1, \pm 2, \cdots$$

  The discrete spectrum can also be represented as:

$$X(f) = \sum_{n=--\infty}^{+\infty} X(n)\delta(f-nf_0)$$

  We can verify that the time function is indeed periodic:

$$x_T(t+kT) = x_T(t+k/f_0) = \sum_{n=-\infty}^{+\infty} X(n)e^{-j2\pi nf_0(t+k/f_0)} = x_T(t)$$

  (for $k = \pm 1, \pm 2, \cdots$)

- **Periodic, discrete time function $x(m)$, discrete, periodic spectrum $X(n)$**

  This is the discrete Fourier transform (DFT).

$$X(n) = \frac{1}{T} \sum_{m=0}^{M-1} x(m) e^{-j2\pi nmf_0t_0}$$

$$n = 0, 1, \cdots, M-1$$

$$x(m) = \frac{1}{F} \sum_{n=0}^{M-1} X(n) e^{j2\pi nmf_0t_0}$$

$$m = 0, 1, \cdots, M-1$$

  where M is the number of samples in the period T, which is also the number of frequency components in the spectrum:

$$M = \frac{T}{t_0} = \frac{1/f_0}{1/F} = \frac{F}{f_0}$$

  We therefore also have $TF = M$ and $t_0 f_0 = 1/M$.

  The DFT can be redefined as

$$X(n) = \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} x(m) e^{-mn\,j2\pi/M} = \sum_{m=0}^{M-1} w_M^{mn} x(m)$$

$$n = 0, 1, \cdots, M-1$$

$$x(m) = \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} X(n) e^{mn\,j2\pi/M} = \sum_{n=0}^{M-1} w_M^{-mn} X(n)$$

$$m = 0, 1, \cdots, M-1$$

  where $w_M \triangleq e^{-j2\pi/M}/\sqrt{M}$.

  We can easily verify that the time function and its spectrum are indeed periodic: $x(m + kM) = x(m)$ and $X(n + kM) = X(n)$.

The δ function

The discrete and periodic time function and spectrum can be written as, respectively

$$x_T(t) = \sum_{m=-\infty}^{+\infty} x(m)\delta(t - mt_0)$$

$$X_F(f) = \sum_{n=-\infty}^{+\infty} X(n)\delta(f - nf_0)$$

The δ function used above satisfies the following:

1.

$$\delta(t-\tau) = \begin{cases} 0 & t \neq \tau \\ \infty & t = \tau \end{cases}$$

2.

$$\int_{-\infty}^{+\infty} \delta(t)dt = 1$$

3.

$$\int_{-\infty}^{+\infty} x(t)\delta(t-\tau)dt = x(\tau)$$

Vector form of 1D-DFT

The above summation expression for DFT can also be written in more convenient form of matrix-vector multiplication:

$$\begin{bmatrix} X(0) \\ \cdot \\ \cdot \\ X(M-1) \end{bmatrix} = \frac{1}{\sqrt{M}} \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & (e^{-j2\pi/M})^{mn} & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x(0) \\ \cdot \\ \cdot \\ x(M-1) \end{bmatrix}$$

and

$$\begin{bmatrix} x(0) \\ \cdot \\ \cdot \\ x(M-1) \end{bmatrix} = \frac{1}{\sqrt{M}} \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & (e^{j2\pi/M})^{mn} & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} X(0) \\ \cdot \\ \cdot \\ X(M-1) \end{bmatrix}$$

It is obvious that the complexity of 1D DFT takes is $O(N^2)$, which, as we will see later, can be reduced to $O(Nlog_2N)$ by Fast Fourier Transform (FFT) algorithms.

These matrix-vector multiplications can be represented more concisely as:

$$\overline{X} = W^{-1}\overline{x}$$

and

$$\overline{x} = W\overline{X}$$

where both $\overline{X}$ and $\overline{x}$ are $M \times 1$ column (vertical) vectors:

$$\overline{X} \triangleq \begin{bmatrix} X(0) \\ \cdot \\ \cdot \\ X(M-1) \end{bmatrix}_{M\times 1}$$

$$\overline{x} \triangleq \begin{bmatrix} x(0) \\ \cdot \\ \cdot \\ x(M-1) \end{bmatrix}_{M\times 1}$$

and W is an $M \times M$ matrix:

$$W = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & w_{mn} & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}_{M\times M}$$

where w_{mn} is an element in the mth row and nth column of matrix W defined as

$$w_{mn} \triangleq \frac{1}{\sqrt{M}}(e^{j2\pi/M})^{mn}$$

whose complex conjugate is

$$w^*_{mn} = \frac{1}{\sqrt{M}}(e^{-j2\pi/M})^{mn}$$

Obviously W is symmetric ($w_{mn} = w_{nm}$)

$$W^T = W$$

but W is not Hermitian:

$$W^{*T} = W^* \neq W$$

W is a unitary matrix,

$$W^{*T} = W^* = W^{-1}$$

because its rows (or columns) are orthogonal:

$$(W_m, W_{m'}) = \sum_{k=1}^{M} w^*_{mk} w_{m'k} = \frac{1}{M}\sum_{k=1}^{M}(e^{-j2\pi/M})^{mk}(e^{j2\pi/M})^{m'k} = \frac{1}{M}\sum_{k=1}^{M}(e^{j2\pi/M})^{(m'-m)k} \stackrel{*}{=} \delta_{m'm}$$

(* Why?)

The DFT pair can be rewritten as:

$$\overline{X} = W^*\overline{x}$$

$$\overline{x} = W\overline{X}$$

Fast Fourier Transform (FFT) Algorithm

The M-point DFT of time samples $x(0), x(1), \cdots, x(M-1)$ is defined as (ignoring the coefficient $1/\sqrt{M}$ for now):

$$X(n) = \sum_{m=0}^{M-1} x(m) e^{-j2\pi mn/M} = \sum_{m=0}^{M-1} x(m) w_M^{mn}$$

for

$$n = 0, 1, \cdots, M-1$$

w_M is defined as $w_M \triangleq e^{-j2\pi/M}$ and it is easy to show that w_M has the following properties:

1. $w_M^{kM} \equiv 1$
2. $w_{2M}^{2k} \equiv w_M^k$
3. $w_{2M}^{M} \equiv -1$

Let $M = 2N$, the above DFT can be written as

$$X(n) = \sum_{m=0}^{N-1} x(2m) w_{2N}^{2mn} + \sum_{m=0}^{N-1} x(2m+1) w_{2N}^{(2m+1)n}$$

The first summation has all the even terms and the second all the odd ones. Due to the 2nd property of w_M, the above can be rewritten as

$$X(n) = \sum_{m=0}^{N-1} x(2m) w_N^{mn} + \sum_{m=0}^{N-1} x(2m+1) w_N^{mn} w_{2N}^{n}$$

We define

$$X_{even}(n) \triangleq \sum_{m=0}^{N-1} x(2m) w_N^{mn}$$

and

$$X_{odd}(n) \triangleq \sum_{m=0}^{N-1} x(2m+1) w_N^{mn}$$

They are $M/2$-point DFTs. The original M-point DFT becomes

$$X(n) = X_{even}(n) + X_{odd}(n) w_{2N}^{n} \tag{1}$$

Here we let the index n cover only the first half of the original range of the DFT, $n = 0, 1, \cdots, M/2 - 1 = N - 1$. The second half can be obtained by replacing n in Eq. (1) by $n + N$:

$$X(n+N) = X_{even}(n+N) + X_{odd}(n+N)w_{2N}^{n+N}$$

Due to the first property of w_M, we have

$$X_{even}(n+N) = \sum_{m=0}^{N-1} x(2m)w_N^{m(n+N)} = \sum_{m=0}^{N-1} x(2m)w_N^{mn} = X_{even}(n)$$

and similarly

$$X_{odd}(n+N) = X_{odd}(n)$$

Also, due to the 3rd property of w_M, we have

$$w_{2N}^{n+N} = w_{2N}^n w_{2N}^N = -w_{2N}^n$$

Now the second half of the DFT becomes

$$X(n+N) = X_{even}(n) - X_{odd}(n)w_{2N}^n \tag{2}$$

The M-point DFT can now be obtained from Eqs. (1), (2), once $X_{ever}(n)$ and $X_{odd}(n)$ are available. However, since $X_{even}(n)$ and $X_{odd}(n)$ are M/2-point DFTs, they can be obtained the same way. This process goes on recursively until finally only 1-point DFTs are needed, which are just the time samples themselves. Therefore, the operations of an M-point DFT can be symbolically represented by the following diagram. The complexity is therefore reduced from $O(M^2)$ to $O(Mlog_2M)$.

Fourier Transform 2 Real Functions with 1 DFT

First we recall the symmetry properties of the DFT. The DFT of $x(m) = x_r(m) + jx_i(m)$ is defined as

$$
\begin{aligned}
X(n) &= \sum_{m=0}^{M-1} x(m)e^{-j2\pi mn/M} \\
&= \sum_{m=0}^{M-1} x_r(m)e^{-j2\pi mn/M} + j\sum_{m=0}^{M-1} x_i(m)e^{-j2\pi mn/M} \\
&= \sum_{m=0}^{M-1} x_r(m)cos(2\pi mn/M) - j\sum_{m=0}^{M-1} x_r(m)sin(2\pi mn/M) \\
&\quad +j[\sum_{m=0}^{M-1} x_i(m)cos(2\pi mn/M) - j\sum_{m=0}^{M-1} x_i(m)sin(2\pi mn/M) \\
&= \sum_{m=0}^{M-1} [x_r(m)cos(2\pi mn/M) + x_i(m)sin(2\pi mn/M)] \\
&\quad +j\sum_{m=0}^{M-1} [x_i(m)cos(2\pi mn/M) - x_r(m)sin(2\pi mn/M)] \\
&= X_r(n) + jX_i(n)
\end{aligned}
$$

where $X_r(n)$ and $X_i(n)$ are the real and imaginary part of the spectrum respectively. If $x(m)$ is real, i.e., $x_i(m) \equiv 0$, then we have

$$
\begin{cases} X_r(-n) = X_r(n) \\ X_i(-n) = -X_i(n) \end{cases}
$$

or

$$
X(-n) = X_r(-n) + jX_i(-n) = X_r(n) - jX_i(n) = X^*(n)
$$

If $x(m)$ is imaginary, i.e., $x_r(m) \equiv 0$, then we have

$$
\begin{cases} X_r(-n) = -X_r(n) \\ X_i(-n) = X_i(n) \end{cases}
$$

or

$$
X(-n) = X_r(-n) + jX_i(-n) = -X_r(n) + jX_i(n) = -X^*(n)
$$

Next we show how an arbitrary function $f(x)$ can be decomposed into the even and odd components $f_e(x)$ and $f_o(x)$:

$$\begin{cases} f_e(x) = (f(x) + f(-x))/2 \\ f_o(x) = (f(x) - f(-x))/2 \end{cases}$$

and

$$f_e(x) + f_o(x) = f(x)$$

Now we are ready to show how to Fourier transform two real functions $x_1(m)$ and $x_2(m)$ to get their spectra $X_1(n)$ and $X_2(n)$ by one DFT.

1. Define a complex function $x(m)$ by the two real functions:

   $$x(m) \triangleq x_1(m) + jx_2(m)$$

   Notice here that we impose j on $x_2(m)$ to make it imaginary.

2. Find the DFT of $x(m)$

   $$DFT[x(m)] = X(n) = X_r(n) + jX_i(n)$$

3. Separate $X(n)$ into $X_1(n)$ and $X_2(n)$, the spectra of $x_1(m)$ and $x_2(m)$, using the symmetry properties discussed previously.

   - Since $x_1(m)$ is real, the real part of its spectrum $X_1(n)$ is the even component of $X_r(n)$ and the imaginary part of $X_1(n)$ is the odd component of $X_i(n)$, i.e.,

     $$X_1(n) = X_{1r}(n) + jX_{1i}(n) = \frac{X_r(n) + X_r(-n)}{2} + j\frac{X_i(n) - X_i(-n)}{2}$$

   - Since $jx_2(m)$ is imaginary, the real part of its spectrum $jX_2(n)$ is the odd component of $X_r(n)$ and the imaginary part of $jX_2(n)$ is the even component of $X_i(n)$, i.e.,

     $$jX_2(n) = j[X_{2r}(n) + jX_{2i}(n)] = \frac{X_r(n) - X_r(-n)}{2} + j\frac{X_i(n) + X_i(-n)}{2}$$

     Dividing both sides by j, we get

     $$X_2(n) = X_{2r}(n) + jX_{2i}(n) = \frac{X_i(n) + X_i(-n)}{2} - j\frac{X_r(n) - X_r(-n)}{2}$$

   Note that $X(-n) = X(N-n)$ because $X(n)$ is a periodic function.

Two-Dimensional Fourier Transform (2D-FT)

Similar to 1D-FT, 2D-FT can also have four different forms depending on whether the 2D signal (usually spatial signal) $f(x,y)$ is periodic and whether it is discrete. Here we consider only two cases:

- 2D Fourier transform pair of a Non-periodic, continuous signal $f(x,y)$ is

$$F(u,v) = \int\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dx\,dy$$

$$f(x,y) = \int\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}du\,dv$$

  where u and v are spatial frequencies in x and y directions, respectively, and $F(u,v)$ is the 2D spectrum of $f(x,y)$.

- 2D discrete Fourier transform pair of a finite (periodic) and discrete signal $x(m,n)$, $(0 \le m \le M-1,\ 0 \le n \le N-1)$ is

$$X(k,l) = \frac{1}{\sqrt{MN}} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} x(m,n)e^{-j2\pi(\frac{mk}{M}+\frac{nl}{N})}$$

$$x(m,n) = \frac{1}{\sqrt{MN}} \sum_{l=0}^{N-1} \sum_{k=0}^{M-1} X(k,l)e^{j2\pi(\frac{mk}{M}+\frac{nl}{N})}$$

$$(0 \le m,k \le M-1, \quad 0 \le n,l \le N-1)$$

  where M and N are the numbers of samples in x and y directions, respectively, and $X(k,l)$ is the 2D discrete spectrum of $x(m,n)$. Both $X(k,l)$ and $x(m,n)$ can be considered as elements in two M by N matrices $[x]$ and $[X]$, respectively.

Example 1

$$f(x,y) = \begin{cases} 1 & if\ (-\frac{a}{2} < x < \frac{a}{2},\ -\frac{b}{2} < y < \frac{b}{2}) \\ 0 & else \end{cases}$$

$$\begin{aligned} F(u,v) &= \int\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx\,dy \\ &= \int_{-a/2}^{a/2} e^{-j2\pi ux} dx \int_{-b/2}^{b/2} e^{-j2\pi vy} dy \\ &= \frac{sin(\pi ua)}{\pi u} \frac{sin(\pi vb)}{\pi v} \end{aligned}$$

See Fig. 3.2 on page 85 of the text book.

Example 2

$$f(x,y) = \begin{cases} 1 & x^2 + y^2 < R^2 \\ 0 & else \end{cases}$$

It is more convenient to use polar coordinate system in both spatial and frequency domains. Let

$$\begin{cases} x = r\,cos\theta, & y = r\,sin\theta \\ r = \sqrt{x^2+y^2}, & \theta = tan^{-1}(y/x) \end{cases}$$

$$dx\,dy = r dr\,d\theta$$

and

$$\begin{cases} u = \rho\,cos\phi, & v = \rho\,sin\phi \\ \rho = \sqrt{u^2+v^2}, & \phi = tan^{-1}(v/u) \end{cases}$$

$$du\,dv = \rho d\rho\,d\phi$$

we have:

$$\begin{aligned} F(u,v) &= \int\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx\,dy \\ &= \int_0^R [\int_0^{2\pi} e^{-j2\pi r\rho(cos\theta cos\phi + sin\theta sin\phi)} d\theta] r dr \\ &= \int_0^R [\int_0^{2\pi} e^{-j2\pi r\rho cos(\theta-\phi)} d\theta] r dr \\ &= \int_0^R [\int_0^{2\pi} e^{-j2\pi r\rho cos\theta} d\theta] r dr \end{aligned}$$

To continue, we need to use 0th order Bessel function $J_0(x)$ defined as

$$J_0(x) \triangleq \frac{1}{2\pi} \int_0^{2\pi} e^{-jx\,cos\theta} d\theta$$

which is related to the 1st order Bessel function $J_1(x)$ by

$$\frac{d}{dx}(x\, J_1(x)) = x\, J_0(x)$$

i.e.

$$\int_0^x x\, J_0(x) dx = x\, J_1(x)$$

Substituting $2\pi r\rho$ for x, we have

$$\begin{aligned} F(u,v) &= F(\rho,\phi) = \int_0^R 2\pi r\, J_0(2\pi r\rho) dr \\ &= \frac{1}{\rho} R\, J_1(2\pi\rho R) \end{aligned}$$

We see that the spectrum $F(u,v) = F(\rho,\phi)$ is independent of angle ϕ and therefore is central symmetric. See the top example in Fig. 3.3 on page 86 of the text book.

Matrix Form of 2D DFT

Reconsider the 2D DFT:

$$
\begin{aligned}
X(k,l) &= \frac{1}{\sqrt{MN}}\sum_{n=0}^{N-1}[\sum_{m=0}^{M-1} x(m,n)e^{-j2\pi\frac{mk}{M}}]e^{-j2\pi\frac{nl}{N}} \\
&= \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} X'(k,n)e^{-j2\pi\frac{nl}{N}}
\end{aligned}
$$

$$(0 \le m,k \le M-1, \;\; 0 \le n,l \le N-1)$$

where

$$X'(k,n) \stackrel{\triangle}{=} \frac{1}{\sqrt{M}}\sum_{m=0}^{M-1} x(m,n)e^{-j2\pi\frac{mk}{M}}$$

As the summation is with respect to the row index m and the column index n can be treated as a fixed parameter, this expression can be considered as the Fourier transform of the nth column of $[x]$, which can be written in column vector (vertical) form as:

$$\overline{X'}_n = W^*\overline{x}_n$$

for all columns $n = 0, \cdots, N-1$.

Putting all these N columns together, we can write

$$\left[\overline{X'}_0, \cdots, \overline{X'}_{N-1}\right] = W^* \; [\overline{x}_0, \cdots, \overline{x}_{N-1}]$$

or more concisely

$$[X'] = W^* \; [x]$$

where W^* is a N by N Fourier transform matrix.

We then notice that the summation expression for $X(k,l)$ is with respective to the column index n and the row index number k can be treated as a fixed parameter, the expression is the Fourier transform of the kth row, which can be written in row vector (horizontal) form as

$$\overline{X}_k^T = (W^*\overline{X'}_k)^T = \overline{X'}_k^T W^{*T}, \quad (k = 0, \cdots, M-1)$$

Putting all these M rows together, we can write

$$\begin{bmatrix} \overline{X}_0^T \\ \cdot \\ \cdot \\ \overline{X}_{M-1}^T \end{bmatrix} = \begin{bmatrix} \overline{X'}_0^T \\ \cdot \\ \cdot \\ \overline{X'}_{M-1}^T \end{bmatrix} W^*$$

(W is symmetric: $W^{*T} = W^*$), or more concisely

$$[X] = [X']\, W^*$$

Substituting $[X']$ by $W^*\ [x]$, we have

$$[X] = W^*\ [x]\ W^*$$

This transform expression indicates that 2D DFT can be implemented by transforming all the rows of $[x]$ and then transforming all the columns of the resulting matrix. The order of the row and column transforms is not important.

Similarly, the inverse 2D DFT can be written as

$$[x] = W\ [X]\ W$$

Again note that W is a symmetric Unitary matrix:

$$W^{-1} = W^{*T} = W^*$$

It is obvious that the complexity of 2D DFT is $O(M^3)$ (assuming $M = N$), which can be reduced to $O(M^2 log_2 M)$ if FFT is used.