Fourier Descriptor — E186 Handout

Definition

Let $x(m)$ and $y(m)$ be the coordinates of the mth pixel on the boundary of a given 2D shape, a complex number can be formed as $z(m) = x(m) + jy(m)$, and the *Fourier Descriptor (FD)* of this shape is defined as:

$$Z(k) = DFT[z(m)] = \frac{1}{N}\sum_{m=0}^{N-1} z(m)e^{-j2\pi mk/N} \quad (k = 0, \cdots, N-1)$$

where N is the total number of pixels along the boundary.

FD can be used as a representation of 2D closed shapes independent of its location, scaling, rotation and starting point. For example, the first $L < N$ FD's $\{Z(k),\ k = 0, \cdots, L\}$ can be used to represent the 2D shape as they reconstruct approximately the shape without the details (corresponding to the high frequency components susceptible to noise):

$$\hat{z}(m) = \sum_{k=0}^{L-1} Z(k)e^{j2\pi mk/N} \quad (m = 0, \cdots, N-1)$$

Properties

- **Translation**

  If the 2D shape is translated by a distance $z_0 = x_0 + jy_0$:

  $$z'(m) = z(m) + z_0 \quad (m = 0, \cdots, N-1)$$

  its FD becomes

  $$\begin{aligned} Z'(k) &= \frac{1}{N}\sum_{m=0}^{N-1}[z(m) + z_0]e^{-j2\pi mk/N} \\ &= \frac{1}{N}\sum_{m=0}^{N-1} z(m)e^{-j2\pi mk/N} + \frac{1}{N}\sum_{m=0}^{N-1} z_0 e^{-j2\pi mk/N} \\ &= Z(k) + z_0\delta(k) \end{aligned}$$

  i.e., the translation only affects the DC component $Z(0)$ of the FD.

- **Scaling**

  If the 2D shape is scaled (with respect to origin) by a factor S:

  $$z'(m) = S\ z(m)$$

  its FD is scaled by the same factor:

  $$Z'(k) = S\ Z(k)$$

- **Rotation**

  If the 2D shape is rotated about the origin by an angle ϕ:

  $$z'(m) = z(m)\ e^{j\phi}$$

  its FD is multiplied by the same factor

  $$Z'(k) = Z(k)\ e^{j\phi}$$

- **Starting Point**

  If the starting point on the boundary is shifted from 0 to m_0:

  $$z'(m) = z(m - m_0)$$

  by time shifting theorem, its FD becomes

  $$Z'(k) = Z(k)e^{-j2\pi m_0 k/N}$$

FD used for shape matching

Given two 2D shapes represented by $\{u(m), \quad m = 0, \cdots, N-1\}$ and $\{v(m), \quad m = 0, \cdots, N-1\}$, respectively, we can determine how similar they are to each other, independent of their location, size, orientation and starting point.

First, the matching problem can be approached in spatial domain by minimizing a distance between the two shapes:

$$d(u,v) \stackrel{\triangle}{=} min_{\{v_0, S, \phi, m_0\}} \sum_{m=0}^{N-1} |u(m) - Sv(m-m_0)e^{j\phi} - v_0|^2$$

If $d(u, v)$ is small enough, u and v are similar.

Solving this problem requires a search for the minimum in a 5-dimensional space (v_0 has both real and imaginary parts), and is therefore very time consuming.

We next consider solving the problem in frequency domain by comparing $U(k)$ and $V(k)$, the FD's of the two shapes. The factor of location (v_0) can be easily eliminated by forcing the DC components $U(0)$ and $V(0)$ to zero, i.e., both shapes are centered at the origin. Now the FD's are

$$U(k) = DFT[\ u(m)\ ]$$

and

$$V(k) = DFT[\ Sv(m - m_0)e^{j\phi}\ ] = SV(k)e^{j(\phi - 2\pi m_0 k/N)} = SV(k)e^{j(\phi - k\beta)}$$

where

$$\beta \triangleq -2\pi m_0/N$$

The distance between U and V is defined as

$$D(U, V) \triangleq min_{\{S,\phi,\beta\}} \sum_{k=1}^{N-1} |U(k) - SV(k)e^{j(\phi + k\beta)}|^2$$

We don't have to search a 3-dimensional space to find $D(U, V)$. Consider an objective function which is to be minimized:

$$\begin{aligned}
J(S, \phi, \beta) &\triangleq \sum_{k=1}^{N-1} |U(k) - SV(k)e^{j(\phi + k\beta)}|^2\} \\
&= \sum_{k=1}^{N-1} [U(k) - SV(k)e^{j(\phi + k\beta)}][U^*(k) - SV^*(k)e^{-j(\phi + k\beta)}] \\
&= \sum_{k=1}^{N-1} U(k)U^*(k) + S^2 \sum_{k=1}^{N-1} V(k)V^*(k) - 2S \sum_{k=1}^{N-1} Re[U^*(k)V(k)e^{j(\phi + k\beta)}] \\
&= \sum_{k=1}^{N-1} |U(k)|^2 + S^2 \sum_{k=1}^{N-1} |V(k)|^2 - 2S \sum_{k=1}^{N-1} w(k)cos(\phi + k\beta + \alpha_k)
\end{aligned}$$

where $w(k)e^{j\alpha_k} \triangleq U^*(k)V(k)$ and $w(k)$ is assumed to be real. Also note that if $z = x + jy$ is a complex number, then

$$z + z^* = (x + jy) + (x + jy)^* = x + jy + x - jy = 2x = 2Re[z]$$

To minimize $J(S, \phi, \beta)$, let

$$\frac{\partial J}{\partial S} = 2S \sum_{k=0}^{N-1} |V(k)|^2 - 2 \sum_k w(k) cos(\phi + k\beta + \alpha_k) = 0 \tag{1}$$

$$\frac{\partial J}{\partial \phi} = 2S \sum_{k=0}^{N-1} w(k) sin(\phi + k\beta + \alpha_k) = 0 \tag{2}$$

$$\frac{\partial J}{\partial \beta} = 2S \sum_{k=0}^{N-1} w(k) k sin(\phi + k\beta + \alpha_k) = 0 \tag{3}$$

Solving Eq. (1) for S, we get

$$S(\phi, \beta) = \frac{\sum_k w(k) cos(\phi + k\beta + \alpha_k)}{\sum_k |V(k)|^2} \tag{4}$$

From Eq. (2), we get

$$\begin{aligned} & \sum_{k=0}^{N-1} w(k)[sin\phi \; cos(k\beta + \alpha_k) + cos\phi \; sin(k\beta + \alpha_k)] \\ = \quad & sin\phi \sum_{k=0}^{N-1} w(k) cos(k\beta + \alpha_k) + cos\phi \sum_{k=0}^{N-1} w(k) sin(k\beta + \alpha_k) = 0 \end{aligned}$$

which yields

$$tan\phi = -\frac{\sum_k w(k) sin(k\beta + \alpha_k)}{\sum_k w(k) cos(k\beta + \alpha_k)} \tag{5}$$

Similarly, from Eq. (3), we can also get

$$tan\phi = -\frac{\sum_k w(k) k sin(k\beta + \alpha_k)}{\sum_k w(k) k cos(k\beta + \alpha_k)} \tag{6}$$

Solving Eqs. (5) and (6) simultaneously, we can obtain β and ϕ, given which, we can solve Eq. (4) for S. Then $D(U, V)$ is obtained.