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Lecture 15: Maxwell's Equations

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Review of Maxwell's Equations

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In this video we're going to discuss Maxwell's Equations.

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E and B Describe Forces on Charges

$$F = q(E + v \times B)$$

The Lorentz Force

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Maxwell's Equations describe how electric and magnetic fields behave, so understanding Maxwell will require revisiting electric and magnetic fields. The simplest way to understand fields is to think of them as a shorthand for how charges move around. That understanding is summarized in the Lorentz force equation pictured on this slide. The equation says that a particle with charge q will experience a force in the direction of an electric field and perpendicular to a magnetic field if the charge is moving.

SKIP As a side-note for the physics-minded, this definition of E and B fields isn't is not super-widely used because you can't actually make a point charge to measure E and B if they're defined this way. Even so, I think it gives good understanding of what E and B do, so we're keeping it.

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Units: E=[V/m], H=[A/m], H is B in a Material

$$D = \epsilon E + P$$

Electric Displacement Field [C/m^2]

Polarization Density (macroscopic dipole moment) [C/m^2]

Electric Field [V/m]

Permittivity [F/m] (8.85 pF/m in free space)

$$V = -\int_{x_A}^{x_B} E \cdot dl$$

$$B = \mu\,(H + M)$$

Magnetic flux density [T]

Magnetization [A/m]

Magnetic field strength [A/m]

Permeability [H/m] (4π/10 μH/m in free space)

$$I = \oint H \cdot dl$$

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We need to talk about the units of E field and eventually B field to cement our understanding of them. Electric field is measured in Volts per meter, which suggests that it's some kind of spatial derivative of voltage.

CLICK That's actually true by definition: voltage is defined as an integral of E field.

CLICK However, the story doesn't end there. E field is only really well defined in vacuum unless we add some additional parameters to represent how E interacts with materials. P in this equation is the polarization density, which represents how charges in a material are arranged. If charges in a material are slightly misaligned, they each create a dipole moment, and the sum of those is the polarization density in this equation. The Polarization Density has the same units as the Displacement Field, and both of them measure how much total charge has been moved by a field The units of that charge displacement are Coulombs per meter squared, which are different from the volts per meter in electric field, so we define epsilon, the permittivity of a material, as a constant that tells us haw much charge the E field moves. Permittivity turns voltage into charge, just like a capacitor, so it has units of farads per meter.

CLICK There's a similar formalism for magnetic fields in materials. B is the magnetic flux density, which is measured in Tesla, and it's most closely related to forces on a charge.

However, we're mostly going to be using H, the magnetic field strength, in our calculations. H is handy because it relates closely to current in circuits rather than forces on charges, and you can see that in its units of amps per meter.

CLICK, like voltage, current can be found by integrating over H fields. In this case over a closed contour of H.

Finally, Magnetization, M, represents magnetic fields that are built into the medium that you are considering, just like polarization, magnetic fields are made of the sum of microscopic magnetic moments. Permeability relates H fields to inductance in the same way the permittivity relates E fields to capacitance, and it tells us how much magnetic flux, the B field, is created per unit of H or M.

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Write Down Maxwell's Equations

$$\nabla \cdot E = \rho/\epsilon$$

$$\nabla \cdot H = 0$$

Del dots: charge and zero

$$\nabla \times E = -\mu \frac{\partial H}{\partial t}$$

$$\nabla \times H = \epsilon \frac{\partial E}{\partial t} + J$$

Del crosses: wave stuff and J associated with H

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So, now that we remember what fields are, you should be perfectly prepared to remember physics class. I'd like you to pause the video and try to write down Maxwell's Equations.

CLICK Here they are! If you're anything like me, you did not find these easy to remember. There's a lot of wild symbols, and the fact that I was taught them in a slightly different form every time I learned them didn't help.

CLICK Here's how I remember them. First, I break them into divergence, "del dot", equations and curl, "del cross", equations. I know there is a divergence and a curl equation for both E and H. Second, I associated charge with E field and current with H field in my head. Those two crutches, combined with some of the results we'll see on the next few slides, let me remember that the divergence for E is related to charge, that the divergence of H has to be zero (because magnetic fields are loops), and that the curl of H is related to J. The remaining time varying terms are part of a wave equation, and I just remember to stick those with the curl equations.

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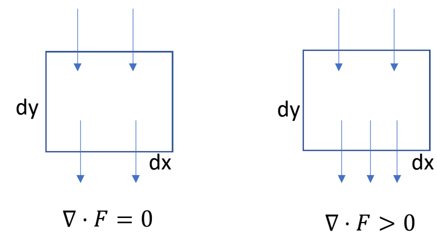
Charge Creates/Destroys E field

$$\nabla \cdot E = \rho/\epsilon$$

$$\nabla \cdot H = 0$$

$$\nabla \times E = -\mu \frac{\partial H}{\partial t}$$

$$\nabla \times H = \epsilon \frac{\partial E}{\partial t} + J$$



Geometric interpretation of (+ve) divergence: more out than in

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These are complicated equations and they have a lot of implications, but I'm going to boil it down to three takeaways for the purpose of this video.

CLICK The first comes from this equation. It says the divergence of the E field is equal to the charge density divided by epsilon.

CLICK Understanding this equation is easiest if you have a geometric picture of the divergence operator. The divergence of a vector field is a differential measure of how much it spreads out. It compares the field flux into a unit volume to the field flux out of it, and if there is extra flux leaving a differential then the field is said to have a positive divergence. So in the example I've shown here, the same amount of field enters and leaves the left box, while more field leaves the right box than came in, which means the field is diverging. This perspective of divergence applied to the first Maxwell equation says that electric field is created by positive charge and absorbed by negative charge.

The other divergence equation says that H field is neither created or destroyed, which means that H field is always a loop, it turns out.

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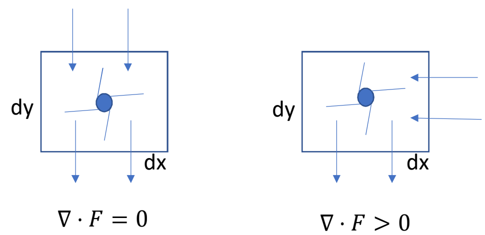
Current Density Implies a Spinning H Field

$\nabla \cdot E = \rho/\epsilon$

$\nabla \cdot H = 0$

$\nabla \times E = -\mu \frac{\partial H}{\partial t}$

$\nabla \times H = \epsilon \frac{\partial E}{\partial t} + J$



Geometric interpretation of (+ve out of page) curl: spins tiny paddle (ccw)

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The second phenomenon I want to highlight is found in this fourth Maxwell's Equation CLICK. If we ignore the time derivative for a minute, we can see that the curl of H field is generated by current density. So current somehow creates magnetic field, or at least a field related quantity.

CLICK a geometric understanding of curl helps understand this implication. Curl can be understood as a differential measurement of how bendy a vector field is. If you could drop a tiny paddle wheel into a vector field, the rate it spins counterclockwise would be related to the curl. In the left figure, our paddle wouldn't spin at all because the left and right blades are pushed the same amount. On the other hand, the right paddle wheel would spin because the vector field must have taken a turn inside that differential which would push favorable on the top blade. So current density creates counterclockwise spinning loops of magnetic field. This fact is why we can find I by integrating around loops of H: if there's a current then there's always going to a magnetic field spinning around it.

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Free Space Equations Imply a Wave

$$\nabla \cdot E = \rho/\epsilon$$

$$\nabla \cdot H = 0$$

$$\nabla \times E = -\mu \frac{\partial H}{\partial t}$$

$$\nabla \times H = \epsilon \frac{\partial E}{\partial t} + J$$

$$\nabla \times \nabla \times E = -\mu \nabla \times \frac{\partial H}{\partial t}$$

$$\nabla \times \nabla \times E = -\mu\epsilon \frac{\partial^2 E}{\partial t^2}$$ Flipped order of derivatives

$$\nabla(\nabla \cdot E) - \nabla^2 E = -\mu\epsilon \frac{\partial^2 E}{\partial t^2}$$ Vector identity + free space

$$\nabla^2 E = \mu\epsilon \frac{\partial^2 E}{\partial t^2}$$ A 3D wave equation!

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The last important takeaway from Maxwell's equations is that they imply the existence of electromagnetic waves.

CLICK First we're going to imagine that we're in free space, which means there is no charge or current lying around to mess with our equations. The two curl equations look like cross coupled differential equations that feature a space derivative on the left and a time derivative on the right. That led to waves last time we checked, so we should see if we can make the magic happen again.

CLICK First we take the curl of both sides of the del cross E equation.
CLICK Then we swap the order of the derivatives and substitute the value of del cross H into the resulting equation.
CLICK We can simplify the left side using a vector identity that we all definitely remember, and then notice that we said del dot E was zero because we're in free space.
CLICK Which finally leaves us with a 3D wave equation! We can pick the velocity of this equation off the coefficient of the time term as usual, and we see that we expect a velocity of 1/sqrt(mu*epsilon).

Summary

- Fields are shorthand for forces
- Charge creates/destroys E
- Current density results in a curling H field
- Free space equations imply a wave with speed $1/\sqrt{\mu\epsilon}$

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Propagating Electromagnetic Waves in Free Space

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In this video we're going to study how the wave we just observed in Maxwell's equations propagates in free space.

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Power Flows Along the Poynting Vector

$S = E \times H$ [W/m^2]=[V/m][A/m], so this measures instantaneous power density (intensity)

$< S > = \frac{1}{2}\mathrm{Re}\{E^* \times H\}$ Analytical representations work here too.

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To talk about how our new found waves propagate, we need to discuss power flow. Power flow for electromagnetic waves is described by the Poynting vector S, which I've always found hilarious because it "points" in the direction of power flow, even though it's not spelled that way. Ahahahhah.

We are skipping the derivation of the Poynting Vector because it's pretty involved, but the upshot is that the instantaneous energy flow is equal to the cross product of E and H fields. This means the direction of power flow is normal to both E and H fields because it's given by the curl between them.

As usual, we care about time averaged power more than instantaneous power. Fortunately, we can use our handy analytical representation to quickly calculate time averaged power flow.

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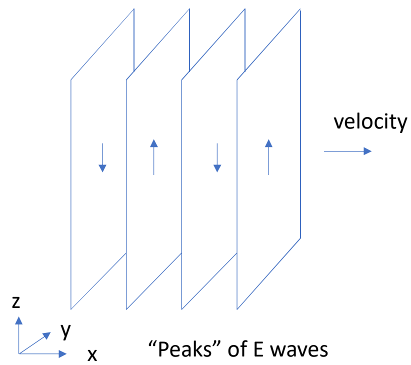
Common to Assume 1D Plane Wave Solution

$\nabla^2 E = \mu\epsilon \frac{\partial^2 E}{\partial t^2}$ Note $\mu = \mu_0$ and $\epsilon = \epsilon_0$

$\frac{\partial^2 E(x,t)}{\partial t^2} = \mu_0\epsilon_0 \frac{\partial^2 E(x,t)}{\partial t^2}$ Assume 1D

$E(x,t) = \hat{z}E_+ e^{j(kx-\omega\ )}$ Sinusoidal “generator” boundary

- E+ Not in $\hat{x}$ direction because propagating that way
- E+ is in $\hat{z}$ direction because it’s easy



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We need to figure out what these waves look like, and for now we’re going to be doing so in free space, so I’ve copied down our wave equation and a reminder that in free space mu is mu_0 and epsilon is epsilon_0.

CLICK the first simplification we’re going to make is to assume that our wave is one dimensional. That makes calculations much easier. It also means that we’re assuming that our waves are planes where there is no variation in the y or z diractions.
CLICK That looks like this. We call this type of 1D wave a plane wave because each moment in time is represented by an entire 2D plane.
CLICK Next we’re going to presume there’s some sinusoidal boundary condition that creates a wave that’s oriented in the z hat direction. You can see our familiar kw-wt formulation inside the wave, indicating that this is a valid solution to the wave equation.
CLICK the choice of having our wave point in the z hat direction might seem weird. That’s because this is a bit arbitrary. We are not letting our E wave point in the direction of propagation because that seems weird (though it’s not impossible, we’ll see more later). If E isn’t pointing in the x hat direction, then we can pick any other direction in the plane arbitrarily because everything is rotationally symmetric for now. We choose z hat because it will make our math easier later.

As a vocabulary note, this z hat coefficient that specifies which way our field is pointing is referred to as a polarization. Our waves are polarized in the z hat direction.

Impedance of Free Space Relates E and H

$E(x,t) = \hat{z}E_+ e^{j(kx-\omega t)}$

$H(x,t) = \hat{y}H_+ e^{j(kx-\omega\ )}$ — H wave equation winds up same, assume in $\hat{y}$ direction for now

$\nabla \times E = -\mu \frac{\partial H}{\partial t}$ — Pick a Maxwell's equation to relate E and H

$$\nabla \times E(x,t) = \hat{x}\left(\frac{dE_z}{dy} - \frac{dE_y}{dz}\right) + \hat{y}\left(\frac{dE_x}{dz} - \frac{dE_z}{dx}\right) + \hat{z}\left(\frac{dE_y}{dx} - \frac{dE_x}{dy}\right) = \hat{y}kE_+ e^{j(kx-\omega t)}$$

$\mu \frac{\partial H(x,t)}{\partial t} = -\mu_0\left(\hat{y}\omega H_+ e^{j(kx-\omega\ )}\right)$ — Correct to assume H wave was in $\hat{y}$ direction! → <u>E normal to H</u>

$\frac{E_+}{H_+} = \frac{\mu_0 \omega}{k} = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377\Omega$ — E and H waves are in-phase, scaled by Impedance of Free Space

Both E and H are normal to propagation, so this is "transverse electromagnetic" or TEM

One question we might have is how E and H are related when they propagate. We're going to find that out by using candidate solutions for E and H. I've copied our E plane wave from an earlier slide, and our H plane wave turns out to be the same form if we chase Maxwell's equations through to find a wave equation for H instead of E. We're going to assume H points in the y hat direction for now, which is a kind of bold assumption. However, if that assumption turns out to be right, then we'll have learned something important about E and H waves, specifically that they are always normal to one another.

CLICK We pick a Maxwell's equation to compare these two candidate solutions. We need one with both an E and an H in it, and this one will do.
CLICK We figure out the left side by remembering for curl after some frantic diving through old notes. The formula turns out to be pretty long. However, it's easy to evaluate because we only have a z component to our E field, and our E field only varies in x, which means that only the dEy/dx derivative produces any quantity. The result points in the y hat direction.
CLICK Next we take the derivative of our candidate H solution to figure out the right hand side of our Maxwell's equation. The result points in the y hat direction still because time derivatives don't affect space, which lines up with the direction of the curl of E. That means our choice of pointing the H field in the y hat direction was correct, it makes the Maxwell's equation self-consistent. So we know that our E field is normal to our H field

because of the curl operators in Maxwell's equations.
CLICK Finally, we can put the results of the past two lines back into the Maxwell's Equation and rearrange the result to find the ratio of E+ to H+, which are the coefficients for our candidate solutions. The result is a constant that is pretty close to 377 ohms, which means E and H fields are always in phase, scaled versions of each other. The amount they're scaled by, this 377 ohm quantity, is called the Impedance of Free Space.
CLICK Finally, one last bit of terminology. E and H point in the z hat and y hat directions, which are both normal to the direction of propagation, x hat. This is a special condition, so this type of wave gets a special name: a transverse electromagnetic, or TEM, wave.

Summary

- S is power flow.

$$< S > = \frac{1}{2}\text{Re}\{E^* \times H\}$$

- E is normal to H. Both are normal to S.
- E and H are in phase, scaled by 377 ohms
- If E and H are normal to propagation, we have a TEM wave

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Electromagnetic Waves in Transmission Lines

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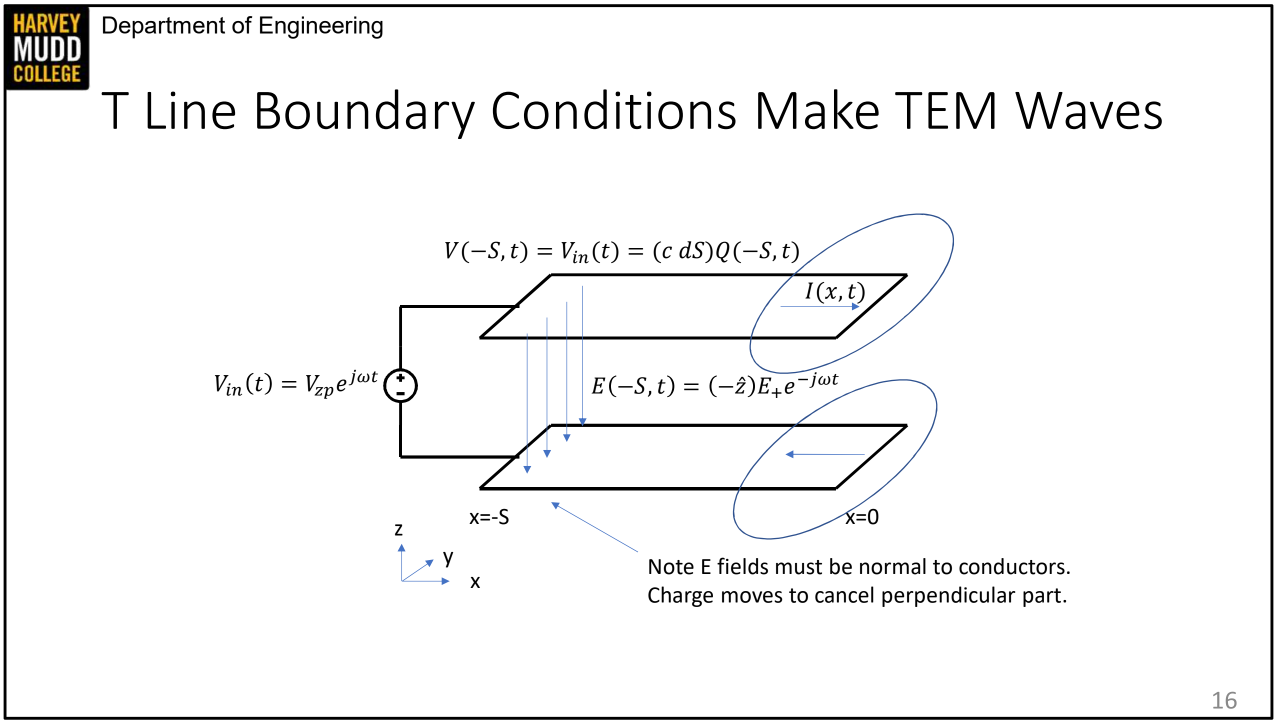
In this video we're going to relate our propagating plane waves to the transmission lines that we know and love.



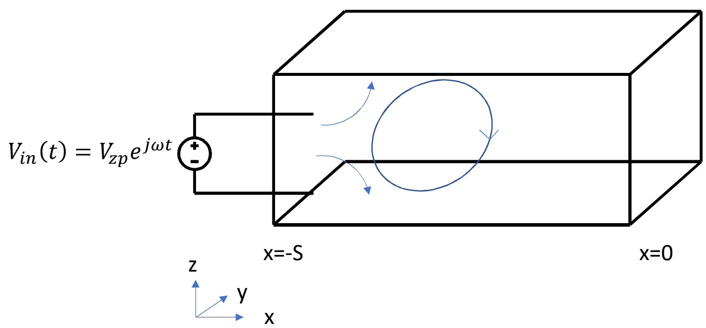
One mystery in our plane wave derivation in the previous video is how we generated the plane wave boundary condition. It turns out that transmission lines are one good answer to that. When we apply a voltage across a transmission line, we shoot a little bit of charge onto it, which creates a vertical electric field in the Z direction. As our voltage varies sinusoidally, so does the field, which creates an E plane wave.

CLICK One thing to observe here is that our E field is always perpendicular to our conductors. That just arises from the geometry of the problem here, but it turns out to always be true that E fields are perpendicular to conductors. That's because conductors are considered to have infinite charge that moves infinitely quickly, so parallel component of an electric field pushing on a conductor would create infinite current, which in turn would result in a charge buildup that cancelled the field. We don't have that problem with perpendicular fields because charges can't jump out of the surface of the conductor, they just build up there to cancel the incident field.

CLICK Current in the transmission line creates an H field curling around it. This means that we have two loops of H field pointing in opposite directions created by the forward and return currents.

These E and H fields are both transverse to the direction of propagation. So this is a TEM wave. The defining feature of transmission lines is that they support the propagation of TEM electromagnetic waves, and that's what separates them from all of the other wires that you've run into.

More Walls Prevent TEM, Called a Waveguide



By way of contrast, if you built a metal rectangle and drove one face of it, you'd have electric fields that pointed in the positive x direction. They'd still propagate down this funny looking container, which would result in moving charges and H fields that formed loops normal to the E field. However, the E and H waves wouldn't both be normal to the direction of propagation anymore, so we wouldn't call them TEM waves. This type of wave container, where the waves aren't TEM, is called a waveguide.

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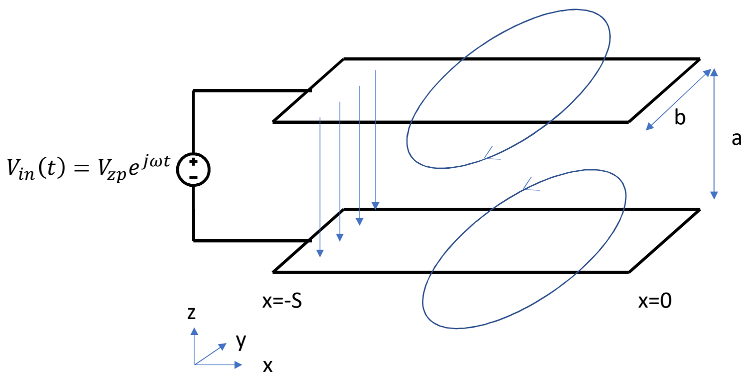
Impedance of T lines ≠ Free Space



$$V = -\int_{x_A}^{x_B} E \cdot dl \qquad I = \oint H \cdot dl$$

$$V(x,t) = aE(x,t) \qquad I(x,t) = bH(x,t)$$

$$\frac{V(x,t)}{I(x,t)} = \frac{a}{b}\frac{E_+}{H_+} = \frac{a}{b}\sqrt{\frac{\mu}{\epsilon}} = Z_0$$

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The impedance of this structure can be calculated by using the relationships between voltage and current. We're going to define a few dimensions for our transmission line, a height a and a width b to show that. We know voltage is the integral of electric field, and because our field is constant between the plates, it's easy to find that V is equal to a times E. We also know that current can by found by a contour integral, and if we integrate in a circle around the transmission line then we find that the upper and lower magnetic fields only add constructively between them, which means that our current is equal to the H field times the width b. Taking the ratio of V over I, we find that we're left with a over B times E+ over H+. We know E+ over H+ is given by the square root of mu over epsilon. That means the impedance of our transmission line is given by a geometric ratio multiplied by the impedance we'd see if our waves were freely propagating in the medium of the circuit board. In short, the geometry of our line, sets our characteristic impedance because it defines how much E and H field our line contains. This is consistent with our previous definition that depended on inductance per unit length and capacitance per unit length, because inductance and capacitance are related to the amount of E and H field stored at one spot in a line.

Summary

- Transmission lines are homes for TEM waves
- Characteristic impedance is set by E and H waves confined by T line dimensions.
- Other types of waves can find homes in waveguides.

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Near-Field Coupling

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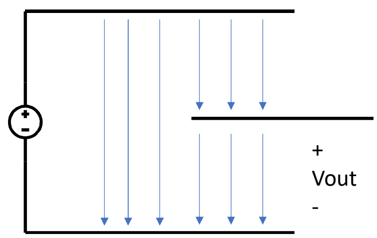
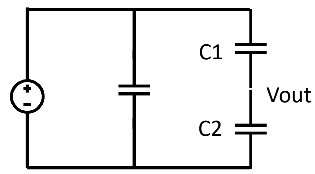
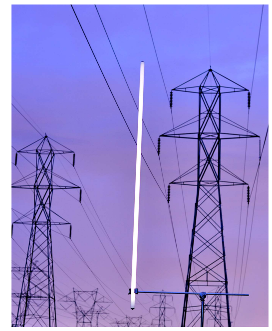
In this video we're going to look at our first examples of wireless power transfer. These will lean on the relations between fields and inductors and capacitors that we saw in Maxwell's equations.

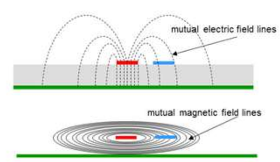
Metal Stick in an E Field Makes Cap Divider







$$V_{out} = \frac{1/j\omega C_2}{1/j\omega C_1 + 1/j\omega C_2} = \frac{C_1}{C_1 + C_2} V_{in}$$



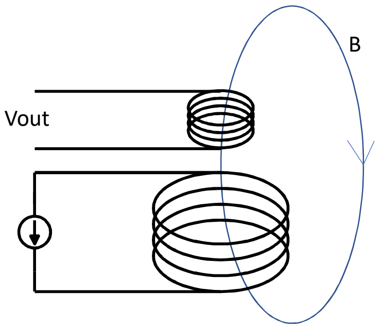
The first kind of easy wireless communication is called capacitive coupling. The picture on the left shows a good model of capacitive coupling: we have put a metal stick in the electric field formed by a a capacitor, and we'd like to know what voltage appears on the stick. Maxwell tells us that field starts and ends on charge, so each set of field arrows must imply a buildup of charge as we apply voltage, in other words, a capacitor.

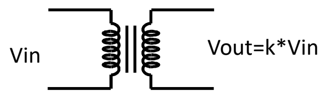
CLICK That means we can represent our stick in a field as a capacitive divider. And we can find the voltage at Vout by writing a standard impedance divider equation. Note that the frequency dependence of impedance falls out of the equation, so this coupling behavior works at almost all frequencies. Though, it is pretty weird at very low frequencies, so this is mostly useful for transmitting or observing high frequency fields.

CLICK There are lots of potential examples of this in the world. I've included a cheeky one where someone left a fluorescent lamp dangling near power lines, and it lit up because of the electric field coupled onto it (though magnetic fields around the lines probably helped with the coupling too). This picture contains one good insight for circuits, which is that your field lines terminate all over the place, especially if you don't take care to build PCBs or breadboards that confine them carefully. In this case, some electric fields from the lines terminate on earth ground. We can see an example fo that in the second picture, where an aggressor trace on a PCB drops some electric field onto a neighboring wire.

This is wireless!
Happens when traces are close together too

Coil in a H field Makes Mutual Inductance





Represent as transformer,
often with low coupling

The second kind of Coupling that we need to discuss is inductive coupling, which arises because of the relationship between current density and the curl of magnetic fields. If we run a current through the lower coil as pictured on the left, then a field must exist in a loop around it. Putting another loop of wire on that field, will induce a current in the second loop of wire. This behavior, where we couple magnetic flux to create a different voltage, is identical to a transformer, and that's how we model it in a circuit. However, because these coils aren't tightly coupled like the coils in a transformer, the coupling between them is given by some coupling constant k instead of the turns ratio N.

The two inductances that make up this transformer are called mutual inductances, because magnetic flux in one will always create magnetic flux in the other.

Inductive coupling is super common, and it's used in almost all near field communication and wireless charging. For example, your wireless tooth brush is inductively charged. You can get parasitic inductive coupling on a PCB if you make a big loop of wire. For instance, a power rail on one edge of top copper on a PCB and a ground rail at the other edge of top copper, will create a big inductive loop that throws out magnetic field.

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Near Field Probes Look Like Loops / Sticks

E probe
Others are B probes

https://www.emrss.com/products/e-h-near-field-probe-set-sniffer-dc-to-9ghz-with-emc-preamplifier

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One way that we measure electric and magnetic fields is by taking advantage of their coupling behavior. I've included a picture of a set of near field probes on this slide. The left most probe measures E field, and it looks like a little stick that you could put in a existing capacitive field. The remaining probes are loops of various sizes that measure B fields. Bigger loops are more sensitive, but less area specific.

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Summary

- Electric fields can carry power/information in capacitive dividers
- Magnetic fields can carry power/information through transformers
- Near field probes measure E and B fields by using this behavior