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Lecture 12.5: Power Flow II

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Input Reflection Coefficient

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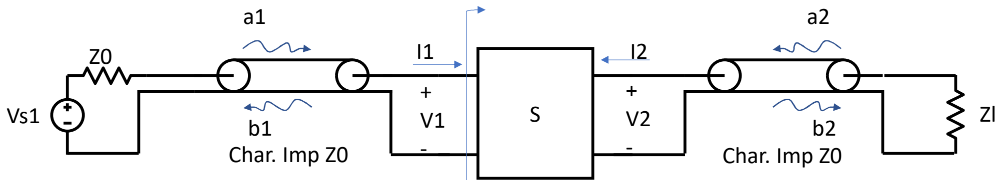
In this video we're going to analyze the effect of unmatched loads on reflections off the input of a two-port network. This is an important lens for analyzing stability because the reflections off of each port of the two-port network are crucial factors in determining stability.

Unmatched Port 1 Reflections Aren't Just S11



$$\Gamma_s = \frac{Z_s - Z_0}{Z_s + Z_0} = 0$$

$$\Gamma_{in} = ?$$

$$\Gamma_l = \frac{Z_l - Z_0}{Z_l + Z_0}$$

$$b_2 = S_{21}a_1 + S_{22}a_2$$

$$a_2 = \Gamma_l b_2$$

$$b_2 = S_{21}a_1 + S_{22}\Gamma_l b_2$$

$$b_2 = \frac{S_{21}}{1 - S_{22}\Gamma_l}a_1$$

$$a_2 = \frac{\Gamma_l S_{21}}{1 - S_{22}\Gamma_l}a_1$$

$$b_1 = S_{11}a_1 + S_{12}a_2$$

$$b_1 = S_{11}a_1 + \frac{S_{12}S_{21}\Gamma_l}{1 - S_{22}\Gamma_l}a_1$$

$$\Gamma_{in} = S_{11} + \frac{S_{12}S_{21}\Gamma_l}{1 - S_{22}\Gamma_l}$$

$$\Gamma_{out} = S_{22} + \frac{S_{12}S_{21}\Gamma_s}{1 - S_{11}\Gamma_s}$$

I've drawn a two-port network here that has a mismatched load. That means that we need to define a reflection coefficient off the load, and we have done so in the form of Gamma_l. The source is matched in this example. We're curious what a reflection off the input port looks like now that we'll see some additional power into port two from load reflections. We'll call the effective reflection coefficient Gamma_in.

CLICK We can start by writing one of the equations that define S parameters, and observing that we need to find b1 over a1. However, to find b1 over a1, we'll need to eliminate a2 from this equation.
CLICK, So we write the other equation that defines S-parameters to do that.
CLICK And we combine that definition with the fact that waves leaving port two get reflected off the load, creating a round trip
CLICK We can substitute that relationship into our equation, CLICK then rearrange it to find b2, CLICK then finally find a2 by multiplying b2 and Gamma_l because a2 is caused by b2 reflecting off the load.
CLICK This gets substituted into our first equations, which leaves us tantalizingly closase to finding Gamma_in.
CLICK factoring out a1 and dividing both sides by it, we find that Gamma_in is given by S11 plus some additional amount that depends on the product of S12, S21 and Gamma_l. I find that product somewhat intuitive because it is the set of reflection coefficients a1 sees to

get back to b1 through port2.
CLICK Finally, note that we could find the reflection behavior of the opposite port by swapping S11 and S22 in the equation.

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There's Another Way to Write Γ_{in}

$$\Gamma_{in} = S_{11} + \frac{S_{12}S_{21}\Gamma_l}{1 - S_{22}\Gamma_l}$$

$$\Gamma_{in} = \frac{S_{11}(1 - S_{22}\Gamma_l)}{1 - S_{22}\Gamma_l} + \frac{S_{12}S_{21}\Gamma_l}{1 - S_{22}\Gamma_l}$$

$$\Gamma_{in} = \frac{S_{11} - \Gamma_l(S_{11}S_{22} - S_{12}S_{21})}{1 - S_{22}\Gamma_l}$$

Determinant of S, called Δ

$$\Gamma_{in} = \frac{S_{11} - \Gamma_l\Delta}{1 - S_{22}\Gamma_l}$$

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There's one common alternate form for Gamma_in that we need to derive.

CLICK We can combine S11 with the second term by multiplying and dividing by the denominator
CLICK, then we factor out Gamma_l and find that it's multiplied by an interesting S parameter quantity.
CLICK This difference of products is the derivative of the S-parameter matrix, which is often given the symbol Delta
CLICK So we can also write Gamma_in in this form, which depends on the S matrix determinant.

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Summary

- The reflection off a port in a S-parameter network depends on mismatch in the load

$$\Gamma_{in} = S_{11} + \frac{S_{12}S_{21}\Gamma_l}{1 - S_{22}\Gamma_l} = \frac{S_{11} - \Gamma_l\Delta}{1 - S_{22}\Gamma_l}$$

Determinant of S

$$\Gamma_{out} = S_{22} + \frac{S_{12}S_{21}\Gamma_s}{1 - S_{11}\Gamma_s} = \frac{S_{22} - \Gamma_s\Delta}{1 - S_{11}\Gamma_s}$$

- This is because of signals taking a round trip, a kind of feedback, which hints that Γ_{in} is important for stability.

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Max Power Transfer with Reflection Coefficients

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In this video we're going to examine what the maximum power transfer theorem says about reflection coefficients of S networks.

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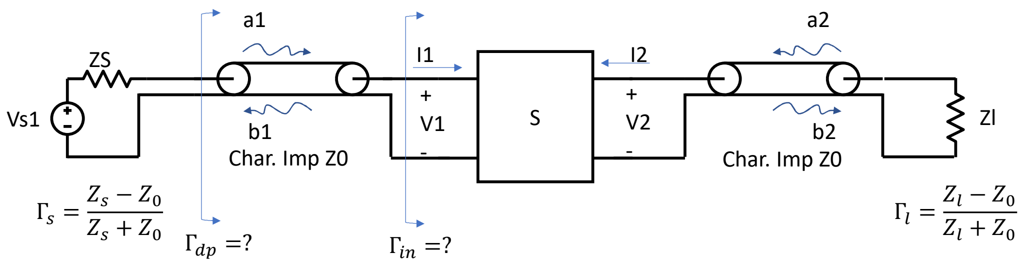
Max Power Transfer Implies Conjugate Γ



Let $Z_{dp} = Z_S^* \rightarrow R_{dp} = R_S, X_{dp} = -X_S$

$$\Gamma_{dp} = \frac{Z_{dp} - Z_0}{Z_{dp} + Z_0} = \frac{(Z_{dp} - Z_0)(Z_{dp} - Z_0)^*}{|Z_{dp} + Z_0|} = \frac{(R_{dp} - Z_0)^2 - X_{dp}^2 + 2j(R_{dp} - Z_0)X_{in}}{|Z_{dp} + Z_0|}$$

$$\Gamma_s = \frac{Z_s - Z_0}{Z_s + Z_0} = \frac{(Z_S - Z_0)(Z_S - Z_0)^*}{|Z_S + Z_0|} = \frac{(R_S - Z_0)^2 - X_S^2 + 2j(R_S - Z_0)X_S}{|Z_{in} + Z_0|}$$

Opposite sign

Therefore: $\Gamma_{dp} = \Gamma_S^* = \Gamma_{in}\exp(2jkS)$

Only zero if $\Gamma_S = 0$

Conjugate matches can have reflections

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Let's start with a standard picture of a S parameter network, but note that I've added an unusual reflection coefficient on the left called Gamma_dp. This is the implied reflection coefficient off the driving point of the left transmission line. We usually think about the driving point impedance, Zdp, but Z and Gamma map to one another 1 for 1, so we can define a Gamma_dp if we feel like it. We're curious about what Gamma_dp will be if we are trying to achieve maximum power transfer.

CLICK Great, to get maximum power transfer we need Zdp to be a conjugate match to Zs, so let's assume that. This has the implication that the resistance RS is equal to RDP, and that the reactance XS is the opposite of XDP.

CLICK We can define Gamma_dp using the reflection coefficient equation, and we're going to want to compare it to Gamma_S in a moment, so I rationalized the denominator then distributed the complex terms in the numerator.

CLICK We can do the same thing with Gamma_S.

CLICK And if we look closely, we can see something interesting about these expressions. The first two terms and the denominator are going to be purely real, and identical, but the imaginary term has the opposite sign between these two expressions.

CLICK Therefore, this driving point impedance is the complex conjugate of the source impedance under max power transfer conditions. We can also rotate our reflection coefficient down our transmission line to find the relationship between Gamma_in and Gamma_S, which is this conjugate match condition plus the phase of the transmission line.

CLICK One interesting detail is that we might have reflections on the line under maximum power transfer conditions. Gamma_in is only zero in this expression if Gamma_S is also zero, so we can have reflections while delivering maximum power. This counterintuitive result has a few explanations. For example, for complex source impedances, we need the line to act like an opposite reactance and cancel out the imaginary part of the source, which requires that we produce a reflected wave with a different phase than an incident one.

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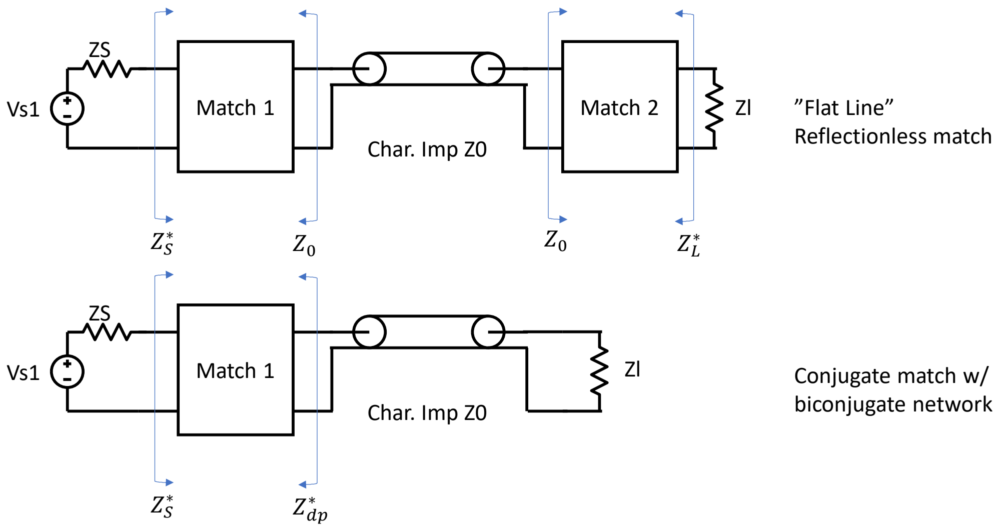
There are Two Ways to Match Systems



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This detail about reflection suggests some interesting design choices you have when you build matching networks, especially if both source and load are mismatched. It is possible to make a reflectionless line by making two matching networks, each matching to Z0 of the line. Alternatively, it is also possible to make a single matching network that matches from ZS to the driving point impedance of a delayed load. These have bandwidth and loss tradeoffs ~ Zdp varies with frequency, which limits the bandwidth of the conjugate match, and designing the biconjugate matching network for two arbitrary impedances can be hard, but having two matching networks on the flat line can increase insertion loss. This hints at the richness of RF design choices, and it's even just the tip of the iceberg because there are may ways to implement each of these matching networks.

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Summary

- Conjugate match between driving point and source implies

$$\Gamma_{dp} = \Gamma_S^* = \Gamma_{in} \exp(2jkS)$$

- Maximum power transfer doesn't mean zero reflections.

- You can design reflectionless systems with two matching networks, or conjugate matched systems with one.

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Power Gain in Terms of S-Parameters

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In this video we're going to derive expressions for power gain using two-port S-parameters. In the last video set we used ideal voltage amplifiers to help us define what power gains were and observe some of their behavior, but we usually use S parameters to characterize two ports, including amplifiers, so it's important that we convert our intuition for power gain into expressions that are usable with S parameters.

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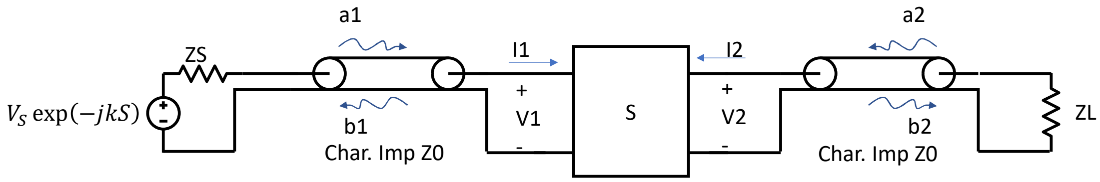
Power Gain Closely Related to S-Parameters



Starting point for Derivation:

$$P_s = \frac{1}{2}|a_1|^2 - \frac{1}{2}|b_1|^2 = \frac{1}{2}|a_1|^2(1-|\Gamma_{in}|^2)$$

$$P_{avs} = \frac{1}{2}|a_1|^2$$

$$P_l = \frac{1}{2}|b_2|^2 - \frac{1}{2}|a_2|^2 = \frac{1}{2}|b_2|^2(1-|\Gamma_L|^2)$$

$$P_{avl} = \frac{1}{2}|b_2|^2$$

Skip to ending point of Derivation:

$$\rightarrow P_s = \frac{|V_s|^2}{8Z_0}\cdot\frac{|1-\Gamma_S|^2}{|1-\Gamma_S\Gamma_{in}|^2}\cdot(1-|\Gamma_{in}|^2)$$

$$\rightarrow P_{avs} = \frac{|V_s|^2}{8Z_0}\cdot\frac{|1-\Gamma_S|^2}{(1-|\Gamma_S|^2)}$$

$$\rightarrow P_l = \frac{|V_s|^2}{8Z_0}\cdot\frac{|1-\Gamma_S|^2}{|1-\Gamma_S\Gamma_{in}|^2}\cdot|S_{21}|^2\cdot\frac{(1-|\Gamma_L|^2)}{|1-S_{22}\Gamma_l|^2}$$

$$\rightarrow P_{avl} = \frac{|V_s|^2}{8Z_0}\cdot\frac{|1-\Gamma_S|^2}{|1-\Gamma_S\Gamma_{in}|^2}\cdot|S_{21}|^2\cdot\frac{(1-|\Gamma_{out}|^2)}{|1-S_{22}\Gamma_{out}^*|^2}$$

With $Z_S = Z_L = Z_0$:

$$\rightarrow P_s = \frac{|V_s|^2}{8Z_0}\cdot(1-|S_{11}|^2)$$

$$\rightarrow P_{avs} = \frac{|V_s|^2}{8Z_0}$$

$$\rightarrow P_l = \frac{|V_s|^2}{8Z_0}\cdot|S_{21}|^2$$

$$\rightarrow P_{avl} = \frac{|V_s|^2}{8Z_0}\cdot\frac{|S_{21}|^2}{(1-|S_{22}|^2)}$$

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We're going to calculate the same useful power quantities we did in the last power gain video, then calculate power gains with them, but we'll use S-parameters instead of real resistances this time. However, we're going to skip a lot of the derivations here because they're not terribly insightful; it's mostly a lot of unexciting algebra relating Gamma values to Z values. I'll link up a full derivation of these powers on my website, but we're going to spend our time trying to get some insight into the physical meaning of each of the terms in the power expressions instead of trying to infer those from a derivation. We're also going to take a look at what these powers look like when ZS and ZL are matched to the lines, because that's usually what you see in lab.

We're going to make some claims about conjugate matches later on this slide, and specifically we're going to say that Gamma_S=Gamma_in* for a conjugate match. That doesn't quite match our result from the previous videos, where we achieved a conjugate match by matching Gamma_S to Gamma_DP. These power gains are usually derived by assuming that the transmission lines are zero length, or, equivalently, that the attached lines are merged into the source Zs so the S network is seeing the driving point impedance of the line and the source together. We're going to follow that simplifying convention and assume the tlines are effectively zero length, so conjugate matches imply Gamma_S=Gamm_in*.

So, starting our analysis with PS:
CLICK The power delivered from our source is the difference between the power in the a1 wave and the power in the b1 wave, and we can express that in terms of Gamma_in if we want to eliminate b1 from the expression.
CLICK We skip the derivation of referring that back to VS, but we can understand what each term in the equation means. The first fraction represents the wave launched onto the line. The second represents some kind of conversion factor between a1 and Vs, which is the tedious math we skpped, and the third reduces PS by the power reflected off of the S parameter input.
CLICK When we match ZS and ZL, the middle term falls away because Gamma_S is zero, and the Gamma_in in the third term is replaced by S11. That's because Gamma_in is made up of S11 plus a round trip signal that bounces of the load, so if Gamma_L is zero Gamma_in reduces to S11. We're left with an interesting expression, which is the incident power multiplied by a factor that captures the power bouncing off the S network input. When S11 is high, we can't deliver power the network, and that is captured by the second term in the matched PS

CLICK The power available from the source is just the a1 wave that it launches.
CLICK We skip the derivation of how to get back from a1 to VS in this expression, but it's worth noting that this constant is very similar to the a1 to VS constant in the PS expression. However, because we can assume Gamma_in is equal to Gamma_s*, the magnitude of Gamma_in squared turns into the magnitude of Gamma_S squared and a bunch of terms cancel.
CLICK When we match the input and the output, we get back to a pure expression of a1.

CLICK The power to the load is given by the power in the b2 wave minus the power in the a2 wave. We can express that in terms of Gamma_L if we want to eliminate a2 from the expression, but we have a long way to go to get from a2 back to Vs.
CLICK Skipping lots of that journey, we find a big expression for PL that is made up of the incident a1 wave, The same a1-to-VS conversion term that we saw in PS up at the top of the page, a gain-looking term that invokes S21 – the forward gain of the network - and a term that captures mismatch with the load.
CLICK When we match ZS and ZL, we see that the input and output mismatch terms fall away, leaving only the incident power multiplied by our gain-like S21 term. This makes some sense: the power delivered to the load is given by the input multiplied by some kind of power gain of the network, and back in our full expression that gain is degraded by source and load mismatch.

CLICK Finally, the power available to the load is just given by the b2 wave.
CLICK Dragging that back through our terrible derivation relates b2 to the driving voltage source. We see that power available to the load is given by the incident power, multiplied by a gain, and multiplied by an output matching term. However, this output matching term differs from Pl because we take advantage of the fact that Gamma_out=Gamma_l* in a conjugate match.

CLICK That Gamma_out term becomes important when we match the input and output. In Pl, the S22 value fell out of the equation because Gamma_l was zero. Here we don't have that guarantee, Gamma_out just becomes S22 because the source termination stops the round trip signal. As a result, we keep a S22 term in our final Pavl expression, which comes out of cancelling the numerator and denominator in this last term of the middle Pavl expression. So this S22 term comes from the act of ideally matching the load; it says "if you have nonzero S22 in your network, that will degrade the power you can deliver, so you will receive a boost to your delivered power in a conjugate match".

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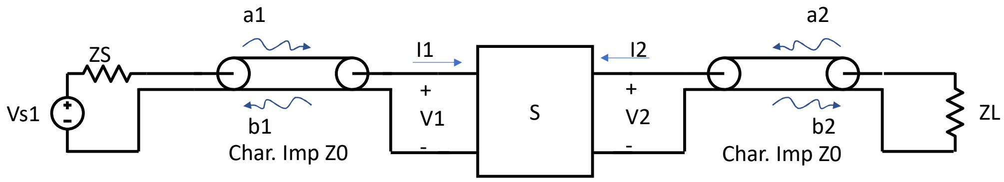
Power Gain Closely Related to S-Parameters



Powers with source and load matched to line:

$$P_s = \frac{|V_s|^2}{8Z_0} \cdot (1 - |S_{11}|^2)$$

$$P_{avs} = \frac{|V_s|^2}{8Z_0}$$

$$P_l = \frac{|V_s|^2}{8Z_0} \cdot |S_{21}|^2$$

$$P_{avl} = \frac{|V_s|^2}{8Z_0} \cdot \frac{|S_{21}|^2}{(1 - |S_{22}|^2)}$$

Power Gains w/ source and load matched to line:

$$G_P = \frac{P_l}{P_s} = \frac{|S_{21}|^2}{1 - |S_{11}|^2}$$

$$G_T = \frac{P_l}{P_{avs}} = |S_{21}|^2$$

$$G_A = \frac{P_{avl}}{P_{avs}} = \frac{|S_{21}|^2}{1 - |S_{22}|^2}$$

Powers from last video

$$P_S = \frac{V_{in}^2}{R_{in}} \left(\frac{R_{in}}{R_s + R_{in}}\right)^2$$

$$P_{avS} = \frac{V_{in}^2}{4R_{in}}$$

$$P_L = \frac{A^2 V_{in}^2}{R_L} \left(\frac{R_{in}}{R_s + R_{in}}\right)^2 \left(\frac{R_l}{R_{out} + R_l}\right)^2$$

$$P_{avL} = \frac{A^2 V_{in}^2}{R_L} \cdot \frac{1}{4} \cdot \left(\frac{R_{in}}{R_s + R_{in}}\right)^2$$

Power gains from last video

$$G_P = A^2 \frac{R_{in}}{R_L} \left(\frac{R_l}{R_{out} + R_l}\right)^2$$

$$G_T = 4A^2 \frac{R_{in}}{R_L} \left(\frac{R_{in}}{R_s + R_{in}}\right)^2 \left(\frac{R_l}{R_{out} + R_l}\right)^2$$

$$G_A = A^2 \frac{R_{in}}{R_L} \left(\frac{R_{in}}{R_s + R_{in}}\right)^2$$

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Whew, that was a lot. I've copied our final, matched powers over to this slide so that we can start figuring out power gains. I've also copied over our power expressions from our previous power gain video, and we're going to compare those to our powers as we go through this derivation.

Let's start with the power gains on the page. There are some really direct comparisons. For instance, Ps is an input power reduced by an input matching term in both cases, and Pavs is just some kind of input power, and it is independent of matching, in both cases. Pl and Pavl are a bit harder to compare. In our DC PL expression, we see that an input power is affected by some gain, then degraded by input and output dividers. However, in our S parameter expression we don't see any input or output matching terms, what gives?

We can look to Pavl for a little insight. Pavl cancels out the effect of a non-ideal S22 value, and it does it by dividing the |S21|^2 gain by a correction factor of (1-|S22|^2). That suggests that for S parameter expressions, we can assume we are ignoring a matching condition if we see a correction factor in the denominator. Returning to the Pl example, we see that the |S21|^2 isn't divided by anything, so we're not "boosting away" the effects of either S11 or S22. That means that this S-parameter PL expression is comparable to the DC PL expression because both are degraded by input and output matching conditions. We can apply similar reasoning to the Pavl expressions – both expressions are affected by input matches, but the S parameter version boosts out the output match while the DC expression replaces it with the optimal, matched factor of ¼.

CLICK Let's consider power gain first. In our S parameter expression, we find that this is |S21|^2 over a correction factor of 1-|S11|^2. We made the claim in the last video that power gain ignores the input match, and we see that in our DC GP because three is no RS divider, it fell out diviing PL by PS. We see the same thing in our S parameter expression, we have a S11 correction factor in our gain, and that's because our denominator, PS, is made smaller by high S11 values. So we boost our gain by 1/(1-|S11|^2) because PS is small. This is the exact same effect as cancelling out the dividers in the DC case.

CLICK For transducer gain, we don't see the same correction factor. That's because Pavs is not reduced by the input match, so we're comparing the power we deliver to the best source power we could receive. This is analogous to our DC GT expression because it forgives neither S11 nor S22, just like DC GT has both input and output dividers.

CLICK Finally, for available gain, we see that we're correcting for S22 but not for S11. This matches our GA expression where we have an input divider and no output divider. In both cases we are correcting for an output match – in the S parameter case it's because we get this 1/(1-|S22|^2) term from conjugate matching our S@@, and in the DC case it's because we assume a matched load.

So, summarizing, we see that there's a clear connection between our S parameter expressions and our DC expressions, and that connection implies that each expression cares about the same thing: GP neglects the input match, GA neglects the output match, and GT accounts for both. We also see the nice behavior that all the gains collapse to the same value, |S21|^2 if the network has a perfect input match, S11=0, and a perfect output match, S22=0.

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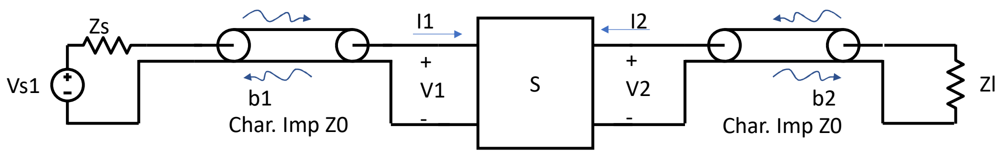
Unilateral Systems Have Simple Gains



$$G_T = \frac{(1-|\Gamma_s|^2)|S_{21}|^2(1-|\Gamma_l|^2)}{|(1-S_{11}\Gamma_s)(1-S_{22}\Gamma_l) - S_{12}S_{21}\Gamma_s\Gamma_l|^2}$$

Let S12=0

$$G_{TU} = \frac{(1-|\Gamma_s|^2)}{|(1-S_{11}\Gamma_s)|^2}|S_{21}|^2\frac{(1-|\Gamma_l|^2)}{|(1-S_{22}\Gamma_l)|^2}$$

$$G_{TU,max} = \frac{1}{1-|S_{11}|^2}|S_{21}|^2\frac{1}{1-|S_{22}|^2} = \frac{P_{avl}}{P_S}$$

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Earlier, we took a shortcut by assuming ZS and ZL were matched to Z0. We can still find power gains if we allow ZS and ZL to take other values, they're just more complicated. The transducer gain here can be derived by dividing our full, unmatch PL and Pavs from two slides ago. The math is uglier, but you can see that this reduces to the magnitude of S21 squared if Gamma_s and Gamma_l are zero. Great!

CLICK Another interesting way to interpret this expression arises in networks where S12 is zero, which are called unilateral networks. S12 being zero makes power only flow form port 1 to port 2, and never from port 2 to port 1. If that's the case, then we can rearrange the terms in the expression GT expression into a unilateral gain GTU. GTU can be factored into three quantities: one that depends only on S11 and Gamma_s, one that depends on the magnitude of S21 squared, and a third that depends only on S22 and Gamma_s. The S11 and S22 quantities are referred to as mismatch factors, and you can picture power trying to get into the S parameter network through the source mismatch, getting scaled by S21, and then trying to get out of the network through the load mismatch.

One takeaway here is that unilateral networks are desirable. At minimum, they make analysis and design easier, but it turns out that they're great for stability, which we'll talk about soon. However, trying to completely squash S12 often has difficult tradeoffs with bandwidth or stability within a network. As a result, much effort is spent getting S12 in just the right spot.

CLICK Finally, If we assume ZS and ZL are conjugate matched to the line, GTU becomes GTU,max the maximum unilateral transducer gain. Which is a nice upper bound on the performance of S networks. Like before, we can also find the max unilateral transducer gain by dividing Pavl by Ps.

Summary

- Power gains are easy to express in S-parameters when source and load impedances are matched to the line

$$G_P = \frac{P_l}{P_S} = \frac{|S_{21}|^2}{1-|S_{11}|^2} \qquad G_T = \frac{P_l}{P_{avs}} = |S_{21}|^2 \qquad G_A = \frac{P_{avl}}{P_{avs}} = \frac{|S_{21}|^2}{1-|S_{22}|^2}$$

$$G_{TU,max} = \frac{1}{1-|S_{11}|^2}|S_{21}|^2\frac{1}{1-|S_{22}|^2} = \frac{P_{avl}}{P_S}$$

- Like the last time we derived power gains:
  - Transducer gain is most pessimistic, other gains "boost out" mismatches
  - Gains reduce to |S21|^2 if S11 and S22 are zero (in/out match lines)