Department of Engineering

Lecture 10: S Parameters

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Two-Port Networks

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In this video we're going to start developing a theory for understanding linear circuits that have two ports, which is a departure from the networks we've looked at up until this point.

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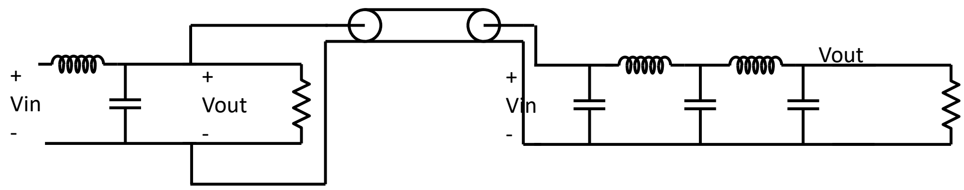
Match Networks and Filters have 2 Ports



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As a reminder, a port is a place where we measure circuits, and in this class we've been pretending we only measure 1 port circuits. Everything we've looked at thus far has been some combination of inductors, capacitors and transmission lines that terminates in a load. But we're straining under that idea, because we spent a lot of time talking about how much power went through a filter into a load. We don't really have numerical tools that describe how much of a signal goes through our circuits, and that's because treating everything as a 1 port network is a deceptive because many circuit really have two ports. CLKICK We can see that really clearly if we take off the loads from the matching network and filter pictured here: which reveals that they have both an input and an output, CLICK and if we were feeling really crazy we could hook one output to another input to make a filter that is matched to a different impedance.

That poses a problem for us, because even though we've analyzed matching networks and filters separately, we have no systematic way to combine the two analyses. We could probably figure something out -- the filter will look like some impedance in its pass band, and we could say that's the load on our matching network -- but in this set of videos we're going to describe a more mathematically rigorous way to talk about linear two port networks.

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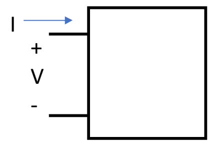
Linearity: Each Port IV Set by Weighted Sum

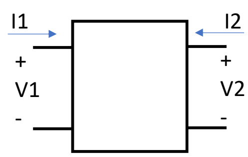
1 Port:



Defined (equivalently) by:

$V = Z_{th} I$ OR $\Gamma = \frac{Z_{th} - Z_0}{Z_{th} + Z_0}$

2 Port:



Each port can affect each other port, so the network is defined by 4 numbers:

$V_1 = Z_{11} I_1 + Z_{12} I_2$

$V_2 = Z_{21} I_1 + Z_{22} I_2$

OR Four Γ-like numbers w/ arbitrary Z0 values

Typically write as Z matrix: $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$ or $\boldsymbol{V} = Z\boldsymbol{I}$

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We'll start making our formalism for two port networks by looking closely at one port networks. I've drawn an example one port here, and it has a pair of port variables V and I, and I is defined as positive going into the port. As long as the circuit inside the one port is linear and passive, we can describe the relation between V and I using a Thevenin impedance. Or, equivalently, we could specify a reflection coefficient for the port, which is a one-to-one function of the impedance.

We had to add a bit of extra information to our definition when we defined the reflection coefficient, which was the idea of Z0. While we know that we mean Z0 to refer to the characteristic impedance of the driving transmission line, the one port model doesn't know anything about what it's connected to, so Z0 seems like some arbitrary constant the designer picks from the one-port's point of view.

CLICK A linear, passive two port network extends this idea. Instead of having one port, it has two, and those two each have names. Creatively, we call them port 1 and port 2. Each port has a port voltage and a port current associated with it, and the port current always points into the port.

Instead of describing the relation between port voltages and currents with one number, like the Thevenin impedance above, we need four numbers. That's because each port can affect both itself and every other port depending on what circuit is inside the two-port box. I'm being a little fuzzy about what it means for a port to affect another port, and we'll get into it in another video. However, we know that everything in the two-port box is linear, so ports affecting one another has to result in a weighted sum of port variables, which is what the equations I'm showing here indicate.

The weights in this type of equation are called Z parameters, and they are written as a letter with two subscripts, the first is the port being affected, the second is the port doing the affecting. So Z12 describes the effect of port 2 on port 1. There are other types of parameters that can describe a two-port network, including a two port version of reflection coefficient called S parameters, so stay tuned. Just like with one ports, we'll probably have to pick a Z0 to use S parameters.

CLICK Finally, we usually write these equations as a matrix to make them more compact. I'm going to try to indicate vectors with bold font to help you identify matrix equation, but sometimes I might just ask you to pick things up from context. For instance, the Z in this matrix equation represents a matrix, even though I don't use any special notation for it.

Summary

- We care about two port theory because we have been making two port networks. eg: filters, matching networks.
- Two port theory lets us describe “through” behavior in addition to “loading / reflection” behavior.
- Each port has an associated I (into the port) and V
- All the port I and V quantities are related by matrix equations.

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Z Parameters

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In this video we're going to look at a specific set of numbers you can use to describe two port networks called Z parameters. These are similar to Thevenin impedances, and they'll serve as an example of two-port analysis to make the idea more concrete.

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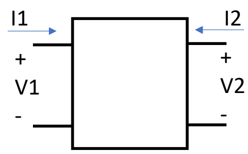
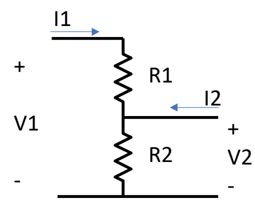
Dividers are Common Examples of Two-Ports



$$V_1 = Z_{11}I_1 + Z_{12}I_2$$
$$V_2 = Z_{21}I_1 + Z_{22}I_2$$



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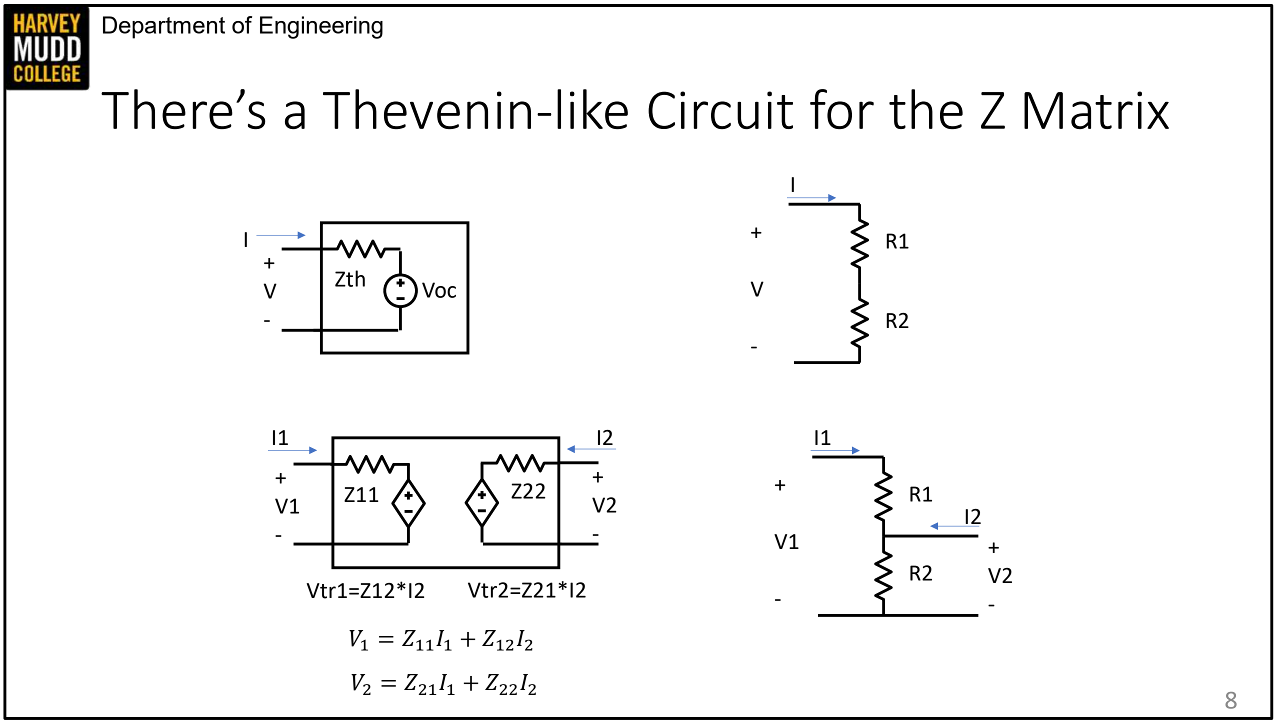
The left side of this slide shows a generic two port while the right side shows a divider, which is a specific example of a two port network. I've drawn ports 1 and 2 on the divider so that we can make a comparison to the generic two port throughout this video. Note that we define all the currents in a two port as pointing into the network for consistency, and we see that in both the generic two port and the divider.

By way of review: I've promised that the voltages of a two port can be described by a weighted sum of the port currents, and I've shown that in a pair of equations here. We called the coefficients of these weighted sums Z parameters.



Any one-port linear network can be described by a Thevenin equivalent circuit. And I've drawn a generic Thevenin equivalent on this slide. Once we find the one-port representation of a linear network, the we can draw either the circuit or its Thevenin equivalent in any schematic we make because the two are electrically identical. So, for instance, the Thevenin impedance of the one-port version of our divider, shown on the right, is R1+R2, and the open circuit voltage, Voc, is zero. (We know Voc is zero because the circuit on the right is passive, and you only get non-zero Voc in active circuits.) That means we can put Zth in place of R1+R2 anywhere and get the exact same I-V behavior.

OK, maybe not so impressive – that's just the definition of series resistors after all – but this in an important analogy for two port networks.

CLICK Two port networks have their own Thevenin-like circuit that you can drop into a circuit in place of a two port. That equivalent circuit is electrically indistinguishable from the resistor divider on the right of the slide, and the heart of two-port analysis is substituting two port models with well defined interaction and loading rules in place of complex circuits. The two port equivalent circuit consists of two resistors, Z11 and Z22, and two current-dependent voltage sources -- Vtr1 and Vtr2, so named after the archaic term "transresistance" -- that are carry port currents across to the other port. The Z12 and Z21 are the control coefficients for these current dependent voltage sources.

CLICK This circuit may seem a little opaque, but it just directly implements the Z-parameter equations we saw on the last slide. It says V1 is a weighted sum of I1 and I2, because the value of the Vtr1 is equal to the Z12*I2 term, and the voltage across Z11 will be Z11*I1.

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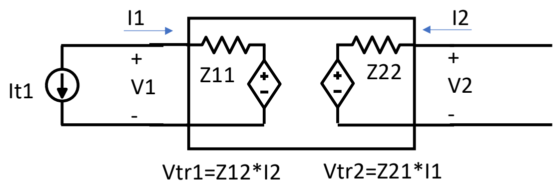
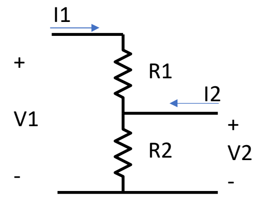
Measure Z Params w/ Test Current + Open Ckt



$$I_1 = I_{t1}$$
$$I_2 = 0$$
$$V_1 = Z_{11} I_{t1}$$
$$V_2 = Z_{21} I_{t2}$$



$$I_1 = I_{t1}$$
$$I_2 = 0$$
$$V_1 = (R_1 + R_2) I_{t1}$$
$$V_2 = R_2 I_{t2}$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} R_1 + R_2 & R_2 \\ R_2 & R_2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

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Now that we have an equivalent circuit, we can think about how to measure or calculate the Z parameters for it.

CLICK The most straightforward way is to drive a current into one port while leaving the other port open circuited. That means we've set the value of I1 to our test current It1, and we've set the value of I2 to zero, because no current flows in an open circuit. We then measure the voltage on port 1, which is all created by the voltage drop on Z11 because I2 is zero, which makes Vtr1 0V accordingly. Using that voltage, we can find Z11. Similarly, if we measure the voltage across port 2, we know that it is all created by Vtr2 because I2 is zero, and there can't be any voltage drop across Z22.

CLICK If we apply this to our resistor divider, we can find its Z-parameters. If I1 is It1, then we can see V1 will be R1+R2 times It1 and V2 will be R2 times It1. That tells us the value of Z11 is R1+R2 and the value of Z21 is R2.

CLICK we can repeat that test on the other port to find the values of Z22 and Z12, and I've summarized them all in the Z-parameter matrix here. We can verify these Z-parameters are right by showing that this matrix gives rise to standard divider behavior: if I2 is zero, which would make this circuit act like a normal voltage divider, then V2 would be I1*R2, V1 would be I1*(R1+R2), and the ratio of V2/V1 would be R2/(R1+R2).

Finally, notice that Z parameters have units of impedance, which is why they're given the symbol Z.

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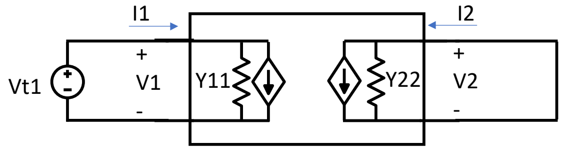
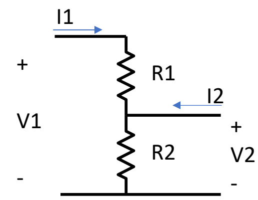
Y Parameters are a 2 Port Version of Norton

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$



$$V_1 = V_{t1}$$
$$V_2 = 0$$
$$I_1 = Y_{11} V_{t1}$$
$$I_2 = Y_{21} V_{t2}$$



$$V_1 = V_{t1}$$
$$V_2 = 0$$
$$I_1 = V_{t1}/R_1$$
$$I_2 = V_{t1}/R_1$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 1/R_1 & 1/R_1 \\ 1/R_1 & 1/(R_1||R_2) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

- To measure: Probe with voltage source, observe with short circuit.

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There are lots of other types of parameters, for instance Y parameters are an important analog to Z parameters. In a Y parameter representation, we represent the port currents as a weighted sum of port voltages, so the Y values are each like conductances. That behavior is captured by a Norton-like equivalent circuit, and we probe its behavior by applying a voltage source to one port and shorting the other port to prevent it from having a voltage value. I encourage you to pause the video and try to find the Y parameters for this divider.

CLICK I've included the answers here.

Other Parameters

- H paramters – "hybrid" parameters, show up on BJT datasheets

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

- G paramters – "inverse hybrid" parameters

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

- All equivalent representations, transforms exist between them.

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Y parameters are popular enough that they deserved special attention, but there are still other sets of parameters that we'll just skim over. H parameters are useful for describing the output current of bipolar junction transistors, so much so that you still see evidence of them on many BJT datasheets. Often the gain parameter on the BJT datasheet is labeled hFE, which stands for "forward emitter h parameter". G parameters also exist, and they are like backwards versions of H parameters that can be used to invert H parameter networks.

One take away from these many transformations is that they are all ways of representing the same circuit. At heart, these are all linear summations of port variables that represent a linear, passive circuit.

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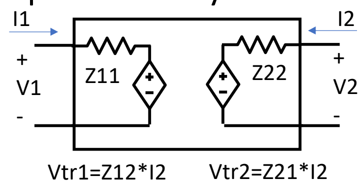
Summary

- Z-parameters are represented by a Thevenin-like model



- You can measure Z parameters with a test current on one port and open circuit loads on all other ports.

- Many other types of parameters exist, notably Y parameters

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S Parameters

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In this video we're going to learn about yet another set of linear parameters called S parameters, where the S is short for scattering. Though we've already got two perfectly good sets of linear network parameters – Y and Z parameters – S parameters are important because they're easy to measure at high frequency. Accordingly, many types of test equipment report S parameter values as their outputs.

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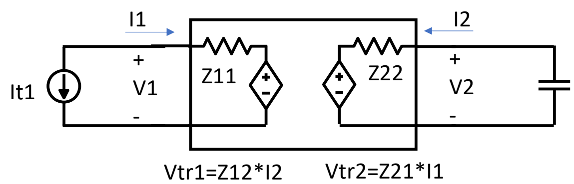
We Can't Measure Z Parameters at High Freq.



- Parasitic capacitances (and antennas) ruin high frequency opens.
- This model neglects transmission line phenomena, fixturing.

We need S parameters because we can't measure Z parameters at high frequency. We established that we measure Z parameters by driving a current source into one port and then measuring voltages. Unfortunately, making a high frequency current source is really tough, and making a high frequency open circuit is almost impossible. CLICK That's because any open circuit we make will look like two pieces of metal sticking out into space, and all pieces of metal have some capacitance between them. Pieces of metal in space also look like antennas, which can cause us trouble too.

Further, this model doesn't account for the transmission lines that we need to hook our equipment up to this two-port circuit, so it doesn't help us understand reflections off of our ports.

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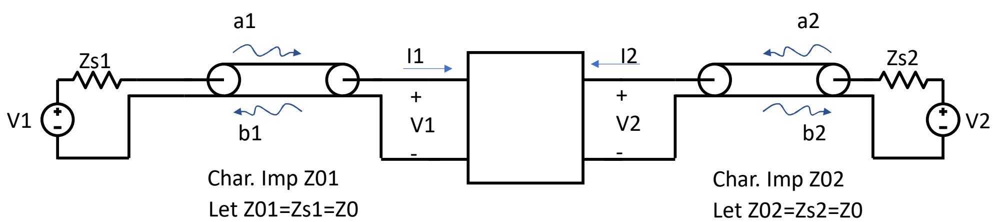
Measuring High-f 2 Ports Requires T Lines



$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

- Could technically have different Z0 on every port.
- S-parameters relate incident and reflected waves a and b.

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S parameters fix these problems by bolting a pair of transmission lines to the ports of the two port. These transmission lines could technically each have different characteristic impedances, and they could each be driven by different source impedances, but we're going to assume that everything is matched to a value Z0 for now.

We're going to assume that each of these transmission lines has a sinusoidal incident wave on it, and we'll say the wave incident on port 1 has amplitude a1, while the wave incident on port 2 has amplitude a2. We'll call the amplitudes of the reflections of these waves b1 and b2. All of these a and b values could be complex in the general case.

The S parameter matrix relates the a and b waves in this model such that the b vector is equal to the S matrix times the a vector. These parameters are called Scattering parameters because they indicate how waves scatter when they hit a network.

This seems like a promising model: S parameters incorporate transmission lines, and we can make pretty good 50 ohm terminations, even at high frequency

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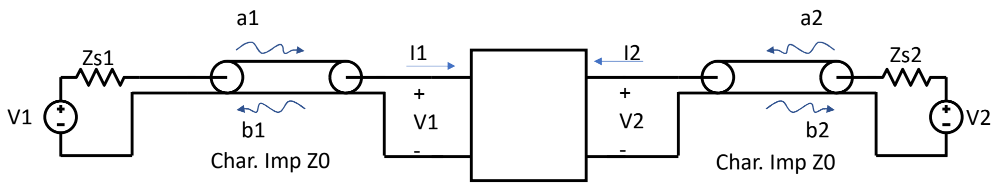
a and b Waves Defined to Calculate Power



$$V_{+1}(x,t) = V_{+1}e^{j(kx-\omega t)}$$

$$P_+(x,t) = \mathrm{Re}\{V_{+1}I_{+1}^*\} = \frac{|V_{+1}|^2}{Z_0}\mathrm{Re}\{e^{2j(kx-\omega t)}\}$$

Analytic representation of power, “complex power”
(Presuming Z0 is purely real here)

$$< P_{+1} >= \frac{|V_{+1}|^2}{2Z_0}$$

Average over sinusoid squared is 1/2

$$a_1 = \frac{|V_+|}{\sqrt{Z_0}} \rightarrow < P_+ >= \frac{|a_1|^2}{2}$$

Pick a definition of a that makes P easy to calculate

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The a and b amplitudes are defined in a special way that makes the calculation of power flow in the system easier.

CLICK So we’ll start to figure out a good value for a by calculating the power carried in our right travelling wave on port 1, V+. I’ve included our complex exponential representation of V+ here.

CLICK We can find the power dissipated at point x of a transmission line by taking voltage times the complex conjugate of current. Because we’re using an analytical representation of voltage, we need to multiply by the complex conjugate of current to make sure our power result turns into a real number. That means our real power will be set a coefficient -- the magnitude of the right-travelling wave squared divided by the characteristic impedance – which I’m assuming is purely real for now – multiplied by a sinusoid.

CLICK If we average the power over time and space to find the average power delivered to the right traveling wave, we get half of that coefficient, because we’re averaging over a sinusoid squared.

CLICK We pick the size of the a wave to easily calculate this coefficient. We set it to the magnitude of the right travelling wave divided by the square root of characteristic impedance. That means the power carried by the wave is given by the magnitude of a^2 divided by 2. We'll circle back to power flow soon, and this calculation will be handy. Note that a can be complex in the general case where Z0 is complex, but its real under our current assumptions.

While we're here, I want to make a note about real and imaginary power. Real power is the this we're used to, electrical energy that gets dissipated as heat. Real power comes from voltage and current that are in phase, and multiplying voltage by the complex conjugate of current is one way to calculate the phase relation between voltage and current. We're going to mostly ignore the imaginary part of the power, but it has an interpretation too: it comes from voltage and current that are out of phase, which means that it corresponds to current and voltage getting stored in inductors and capacitors, then released again during each cycle.

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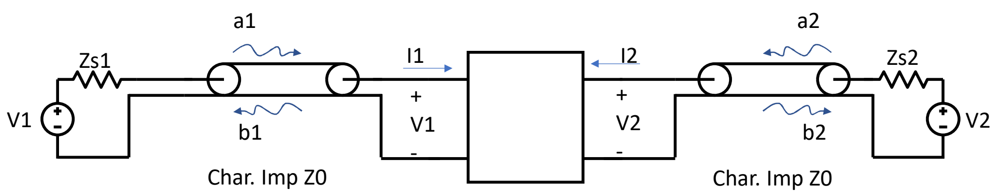
Measure S-Params by Terminating Ports in Z0



Let Zs2=Z0
and V2=0

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$b_1 = S_{11}a_1 + S_{12} \cdot 0$$
$$b_2 = S_{21}a_1 + S_{12} \cdot 0$$

→

$$S_{11} = b_1/a_1 = \Gamma_1 \Big|_{2\ terminated}$$
$$S_{21} = b_2/a_1 = forward\ gain$$
$$S_{12} = b_1/a_2 = reverse\ isolation$$
$$S_{22} = b_2/a_2 = \Gamma_2 \Big|_{1\ terminated}$$

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We measure S parameters by terminating one port in Z0 and turning off the wave going into that port. In other words, CLICK we let Zs2=Z0 and V2=0. That means there will be no a2 wave incident on port 2, so we know reflections b1 and b2 both have to be caused by a1. CLICK this is that statement in equation form: the a2 terms in both of our linear S-parameter equations have been set to zero. CLICK this lets us observe some interesting things about S parameters. S11 is the ratio of b1/a1, which is the ratio of the reflected wave over the incident wave. We already have a name for that, and it's the reflection coefficient. So S11 is the reflection coefficient when port 2 is terminated. S22 is similar: it's the reflection coefficient off of port 2 when port 1 is terminated. S21 and S12 measure types of gain in our system: S21 measures how much of an incident wave crosses from port 1 to port 2, and the often undesirable S12 discusses how much of a wave incident on port 2 makes it back to port 1.

Summary

- We need S-parameters because they're easy to measure at high f
- S-parameters are ratios of incident and reflected waves

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

- a and b waves are sized to make power calculation easy: $a_1 = \frac{|V_+|}{\sqrt{Z_0}}$
- S11 is the reflection coefficient (if port 2 is terminated)

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Calculating S-Parameters in Circuits

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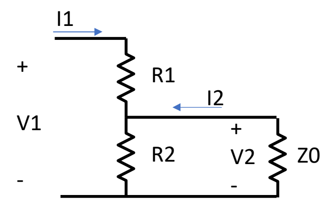
In this video we're going to try to find the S-parameters of a simple circuit.

S-Parameters Don't Map Cleanly to Circuits

$$S_{11} = \Gamma_1 \Big|_{2\ terminated} = \frac{(R_1 + R_2||Z_0) - Z_0}{(R_1 + R_2||Z_0) + Z_0}$$



$$S_{21} = \frac{b_2}{a_1} = \cdots$$

While S-parameters are easy to measure, we're going to find they are tough to analyze. If I was trying to find the S-parameters of our divider, I could start by attaching Z0 to port 2. Making that connection ignores the transmission line connecting port 2 to the termination, but this is OK because the driving point impedance of a transmission line with a matched load is always Z0 anyway. So we can find reflection coefficient of this structure from the Thevenin impedance of the load.

That was fine, but when we have to find S21, we're kind of stumped. S21 is defined in terms of wave amplitudes, but all we have in this model are port voltages and currents, which aren't' the same as the amplitudes of incident and reflected waves.

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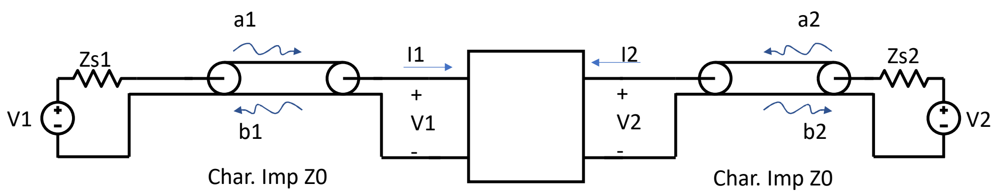
a and b Waves Can Be Related to Port I and V



$$V_1 = V_+ + V_- = \sqrt{Z_0}(a_1 + b_1)$$

$$I_1 = \frac{V_+}{Z_0} - \frac{V_-}{Z_0} = \frac{(a_1 - b_1)}{\sqrt{Z_0}}$$

$$a_1 = \frac{V_1 + Z_0 I_1}{2\sqrt{Z_0}}, \quad b_1 = \frac{V_1 - Z_0 I_1}{2\sqrt{Z_0}}$$

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However, we know the port voltages and currents are given by sums of voltage waves and differences of current waves from our earlier study of transmission lines. CLICK That means we can write a1 and b1 in terms of V1 and I1 by adding and subtracting the V1 and I1 equations in the first two lines.

As an aside: I found these definitions baffling when I first studied S parameters, because they're often the first definitions given for a1 and b1, with no mention of transmission lines. I mentioned earlier that you can think of S parameters without a transmission line around, and pick Z0 to be some arbitrary value, and many texts opt to do so. However, now that I have the picture of S-parameters as describing a two-port with transmission lines bolted on the side, I make sense of these definitions as mathematical tricks to relate the incident and reflected waves to voltage and current in the load.

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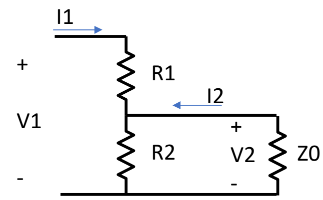
S-Parameters Don't Map Cleanly to Circuits

$$S_{11} = \Gamma_1 \big|_{2\ terminated} = \frac{(R_1 + R_2||Z_0) - Z_0}{(R_1 + R_2||Z_0) + Z_0}$$



Z0*I2=V2, see schematic

$$S_{21} = \frac{b_2}{a_1} = \frac{V_2 - Z_0 I_2}{V_1 + Z_0 I_1} = \frac{2V_2}{V_1\left(1 + \frac{1 - S_{11}}{1 + S_{11}}\right)} = \frac{V_2}{V_1}(1 + S_{11}) = \frac{R_2}{R_2 + R_1}(1 + S_{11})$$

Independent of network, useful takeaway!

Z0/Zl, so that V1=I1*Zl

- Tricky to compute with S-parameters … but doable now.

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So, now that we're armed with these port identities we can get one step farther with finding S21. CLICK the step after requires some fancy circuit footwork. First, we can see from the schematic that I2*Z0 is going to be equal to –V2 because I2 flows into the negative terminal of Z0. So that means we can substitute –V2 for Z0*I2 in the numerator, which means we wind up with a total of 2V2 on top of the expression. The bottom of the expression requires even more tortured reasoning, where we express the normalized load impedance in terms of S11. CLICK However, simplifying this, we get a nice expression that relates the voltage gain of the network to S21, and because we haven't put in any details of this particular divider yet, this expression is good for any two-port network. It's a handy tool, keep track of it.

CLICK Finally, we substitute in the resistor divider equation for V2/V1, in order to find the S21 of this divider.

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Another Option is S-to-Z Conversion Formulas

$$S = (Z - Z_0 I)(Z + Z_0 I)^{-1}$$

$$Z = (I - S)^{-1}(I + S)Z_0$$

$$S_{21} = \frac{2Z_0 Z_{21}}{(Z_{11} + Z_0)(Z_{22} + Z_0) - Z_{12}Z_{21}}$$

- Easy to find Z parameters, and all linear representations equivalent

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If that all seems like a bit much, there's another option. Z parameters were pretty easy to calculate for the divider, and there's a conversion between S and Z parameters. I've included it on this slide. It's a bunch of matrix math, but it lets you do easy circuit analysis to build intuition, and then let a computer do the hard matrix calculations for you. This is a reasonable approach if you don't need to calculate S parameters in your head right now. I've included the specific calculation that relates S21 to Z parameters on this slide to give some context for how annoying these calculations would be to do by hand.

Before we leave the slide, I find it cute that the matrix versions of these equations kind of look like the scalar equations relating reflection coefficient and load impedance: you can see a Zl-Z0 divided by Zl+Z0 in the first equation if you squint.