

Lecture 07: Impedance Matching 1

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E157 – Radio Frequency Circuit Design

Matching Networks

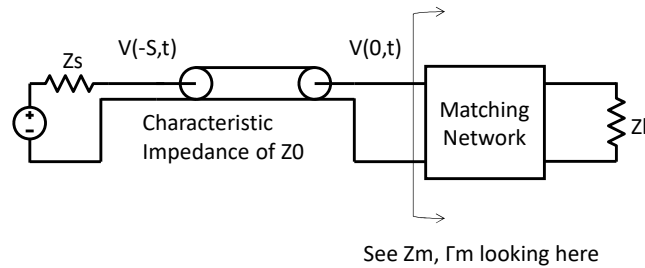
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In this video we're going to start using Smith Charts to make unruly loads behave well at RF frequencies. We're also going to discover that we need to have a good handle on second order RLC circuits to do that.

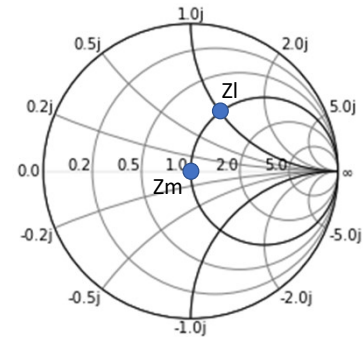
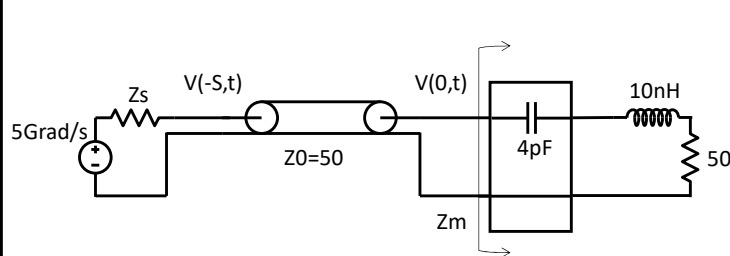
Matching Networks Make Loads Look Like Z_0



- Can we pick a circuit that makes Z_m look like Z_0 for any Z_l ?
- Equivalently, can we pick a circuit to make $\Gamma_m=0$?

The idea of building circuits to make loads behave better is captured by the schematic I've drawn here. I've added a box in between the load impedance and the transmission line which represents a circuit we will design that's called a matching network. We've seen that adding components in shunt or series changes where a load is represented on a Smith Chart, and we're going to use that fact to make loads look like 50 ohms by adding the right shunt and series components. We're calling the impedance looking into a matching network Z_m , and we're calling the reflection coefficient looking into a matching network Γ_m . We're trying to put a circuit in the matching network box that will set Γ_m to zero or, equivalently, set the value of Z_m equal to Z_0 .

If $R_n=1$, Cancel X_n with Opposite Component



$$Z_{nl} = \frac{1}{50} (50 + j(5G)(10n)) = 1 + j$$

$$Z_{ml} = \frac{1}{50} (50 + j(5G)(10n) + 1/(j(5G)(0.004n))) = 1 + j - j$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(4p)(10n)}} = \frac{1}{\sqrt{4 \times 10^{-20}}} = 5 \text{ Grad/s} \quad \leftarrow \text{This is called "ringing out" because L and C resonate at } \omega$$

So let's consider one of the simplest examples of matching networks. Here our load has some inductance in series with it, maybe from a poorly designed via that connects it to ground. We're going to put our unmatched load on the Smith Chart, which, as a reminder, we do that by calculating R_n and X_n . CLICK So we divide both the resistance and reactance of the load by Z_0 and find R_n is equal to 1 and X_n is also equal to 1. Note that I'm using some goofy notation here for the drive frequency and L, where I write the unit prefixes Giga as G and nano as n, which means G times n is one.

This Z_n value is promising. Because $R_n=1$, we almost have a good match, we just need to take care of the non-zero X_n with our matching network. Fortunately, capacitors have negative reactance, so we might consider a network that looks like this CLICK to cancel out the positive reactance from the inductor. I've picked this value of capacitance strategically, and we can see why by computing the normalized match impedance, Z_m CLICK. It's given by almost the same expression as the normalized load impedance, but we have to add the capacitor's impedance in series, and we find the capacitor reactance exactly cancels the inductor reactance. That means Z_m is 50 and Γ_m is 0 as we desired. We can see that on the Smith Chart too: adding the capacitor has moved the load along the $R_n=1$ circle to the origin of the Γ plane.

This is great! We achieved our goal! However, you might wonder whether this network

will work at any other frequency. Answering that question requires us to take a close look at the load we've made, which is a series RLC network. You might recall that series RLC networks are governed by second order differential equations that often have interesting behavior, including resonance. I mention this because we're relying on resonance in this matching network: if we calculate the resonant frequency of it, we discover that it's the same as the drive frequency. Because of that property, this technique of cancelling a reactance with an opposite value is referred to as "ringing out" an impedance.

As a final note, it's worth noting that you can ring out shunt components too. You can just write an admittance equation instead of an impedance to find a very similar set of equations. Also, just as a bit of vocabulary, it's common to refer to an RLC circuit as a tank.

Summary

- Matching networks are circuits we design to make loads look like Z_0 .
- You can cancel out series (or shunt) reactance by adding the other component in series (or shunt).
- Cancelling impedances relies on resonance, which has implications for bandwidth and other performance parameters.

Second Order RLC Circuits, Resonance & Quality Factor

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In this video We're going to take a closer look at RLC circuits because their behavior is so central to matching networks.

ω_0 and Q Describe RLC Circuit Behavior

$$Z_{RLC} = R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right) \quad \text{Resonance when } X=0$$

$$\text{Let } \omega_0 = \frac{1}{\sqrt{LC}} \quad L\omega_{res} = \sqrt{\frac{L}{C}}, \quad C\omega_{res} = \sqrt{\frac{C}{L}}, \quad \text{Let } Z_{0LC} = \sqrt{\frac{L}{C}}$$

$$|Z_{RLC}| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \quad \text{Z magnitude minimized at resonance}$$

$$I_{res} = \frac{V}{R} \quad \text{so } V_L = Z_L I_{res} = j\omega_0 L \frac{V}{R} = jV \frac{\sqrt{L/C}}{R}, \quad \text{Let } Q = \frac{\sqrt{L/C}}{R}$$

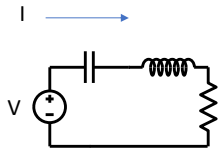
One easy place to start analyzing the behavior of a series RLC circuit is its impedance. I've written the input impedance of an RLC network, Z_{RLC} here. We can tell just looking at it that there's some frequency where Z_{RLC} is purely real, so we can start defining some interesting properties. **CLICK** First, we call the frequency where the imaginary part is zero ω_0 , the natural frequency. We'll also call this the resonant frequency, even though that's a slight misnomer in circuits with a lot of loss. **CLICK** We can check that I calculated ω_0 correctly by calculating $\omega_0 * L$ and $1 / (\omega_0 * C)$, which are both the square root of L / C . This quantity shows up a lot, so it's often referred to, confusingly, as the characteristic impedance of a RLC circuit. We're not going to use that term much more in this class, but I've given it a different symbol than the characteristic impedance of a transmission line just in case. Finally, note that the reactance changes sign above and below resonance, so the system looks a little inductive above resonance and a little capacitive below resonance.

CLICK we can calculate the magnitude of the impedance of this network, and we can see that the magnitude is minimized at resonance. So we're going to get the most current delivered to our resistor at resonance. We also know from our first set of equations that the current delivered at resonance is purely real, so it will be in phase with the voltage V . This is because the reactive components have a net zero impedance at resonance, so it's common to say "series LC tanks ring short" because of the cancellation effect between the inductor and the capacitor.

CLICK That calculation is great, but it overlooks some crazy things that are happening in the inductor and capacitor at the same time. We know the current at resonance is going to be given by V/R because the total reactance is zero, but when we substitute that current into the expressions for inductor voltage, we see that it's 90 degrees out of phase with the resistor voltage and scaled by a factor of the characteristic impedance of the RLC circuit over the resistance. We're going to call that factor Q , or the quality factor. Q is often much greater than one, so a 1V peak-to-peak signal on the resistor can easily cause 10 volts peak-to-peak on the capacitor, so be mindful of voltage and power handling ratings!

It might seem crazy that we have this big voltage running around in our system but it doesn't appear over our resistor at all. CLICK However, if we calculate the capacitor voltage we find that it's also amplified by Q and shifted by negative ninety degrees. The inductor and capacitor voltages exactly cancel, leaving only the real resistor voltage.

Q Describes the Bandwidth of Series RLC



$$V_R = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

$$|V_R| = \frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

If $\pm(\omega L - 1/\omega C) = R$ then power is halved. Quadratic eqn.

$$-\omega_{3dB}L + \frac{1}{\omega_{3dB}C} = R$$

$$\frac{R}{2L} = \frac{R}{2\sqrt{L/C}\sqrt{LC}}$$

$$\omega_{3dB} = \frac{RC \pm \sqrt{(RC)^2 - 4LC}}{2LC} = \frac{\omega_0}{2Q} \pm \sqrt{\left(\frac{\omega_0}{2Q}\right)^2 - \omega_0^2} = \frac{\omega_0}{2Q} \left(1 \pm \sqrt{1 - \frac{1}{4Q^2}}\right)$$

$$\frac{\Delta\omega_{3dB}}{\omega_0} = \frac{1}{2Q} \left[\left(1 + \sqrt{1 - \frac{1}{4Q^2}}\right) - \left(1 - \sqrt{1 - \frac{1}{4Q^2}}\right) \right] \approx \frac{1}{Q}$$

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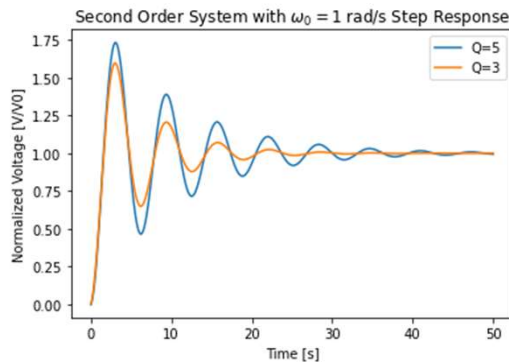
Our motivating question when asking about quality factor was finding the bandwidth of a series RLC circuit. That quantity is very tightly related to the quality factor of the system, such that high quality systems have low bandwidth and low quality systems have high bandwidth.

We can see that by finding the voltage across the resistor using a voltage divider as shown here. CLICK if we find the magnitude of the voltage, we can observe that it will be reduced by the square root of 2 when plus or minus the reactance is equal to the resistance. Reducing voltage by the square root of two halves the power dissipated in the resistor, which is how we get the traditional -3dB definition of bandwidth. CLICK That relation gives rise to a quadratic equation, and I've shown it and the solution here. CLICK Converting from Ls and Cs to omega noughts and Qs requires some creative substitution as shown, but after some aggressive factoring you get a nice relationship that shows the upper and lower 3dB frequencies. Those two frequencies, which fully specify the bandwidth of the RLC circuit, are a Q dependent fraction of the natural frequency.

CLICK We find our bandwidth by looking at the difference between the upper and lower 3dB points, which we call delta omega. In this line we divide the result on the previous line by omega nought and subtract the upper 3dB frequency from the lower one. CLICK The last assumption we make is that Q is kind of big, which allows us to neglect these two

terms for a clear relation: our bandwidth as a fraction of the natural frequency is $1/Q$. This type of description of bandwidth is often called fractional bandwidth.

It Takes $\sim Q$ Cycles for a Transient to Die Out

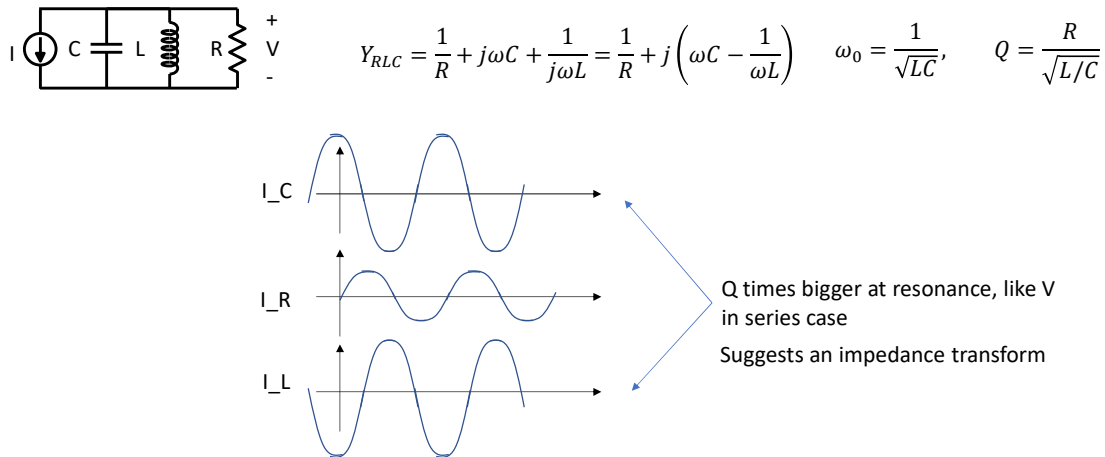


$$V(t) = V_0 \left(1 - \exp\left(-\frac{\omega_0}{2Q}t\right) \cos\left(\omega_0\left(1 - \frac{1}{4Q^2}\right)t\right) \right)$$

- Each cycle takes $T \approx 2\pi/\omega_0$, so Q cycles $\rightarrow V_0 \cdot \exp(-\pi) \cdot (\text{a phase})$
- Similar result: amplitude decay from peak to peak is $\sim \exp(-\pi/Q)$

Because our bandwidth is related to Q , you might expect a handy time domain interpretation of Q as well. Q affects rise time in some interesting ways, but my favorite stupid Q trick is that you can get an approximate measure of Q by counting how many cycles an RLC circuit rings through before it settles to about 5%. You can see this because the envelope of the decaying exponential in a step response is set by the natural frequency and Q . If you set the time t to be one period of 2π over ω_0 , the argument of the exponential becomes $-\pi/Q$. That's handy, but not nearly as handy as remembering that after Q cycles the envelope will have shrunk to e to the minus π , which is about 4% of the initial step. Humans can usually eyeball that value pretty well, so you can pick Q off of second order transient responses just by counting peaks. This slide has some example transient responses for you to practice with.

Parallel RLC is Almost the Same as Series



Finally, we've been focusing on series RLC circuits in this video, but parallel RLC circuits are a natural dual to them. You find the input admittance instead of the input impedance to define resonance, and you drive them with a current source instead of a voltage source because a voltage source would just see three, fairly-boring, independent branches. The natural frequency is calculated the same way as a series RLC circuit, but the quality factor is the inverse. We'll talk about an easy way to remember these quality factors soon.

Like in the series RLC case, the L and C branches of this circuit experience very high stresses at resonance. In this case, the currents in the L and C branches are Q times higher than the R branch, but they are in opposite directions so they sum to zero and the resistor doesn't perceive the big currents running around it in the system.

These big currents suggest an interesting piece of physics. The impedance in the L and C branches has been reduced by a factor of Q compared to the resistor, which is similar behavior to quarter wave transmission lines or various types of transformers, all of these circuits appear to reduce impedance. We're going to describe these impedance transformations soon using a mathematical construct called a series-to-parallel transformation, and we're going to make use of this impedance transformation to build matching networks for systems where R_n is not equal to one. Stay tuned!

Summary

- Impedance is real at resonance
- L and C experience large V (or I) in series (or parallel) resonance
- Natural frequency and quality factor describe 2nd Order systems

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q_{RLCseries} = \frac{\sqrt{L/C}}{R}$$

$$Q_{RLCparallel} = \frac{R}{\sqrt{L/C}}$$

- Q sets the bandwidth of second order band pass: $\frac{\Delta\omega_{3dB}}{\omega_0} \approx \frac{1}{Q}$

Component Quality Factor

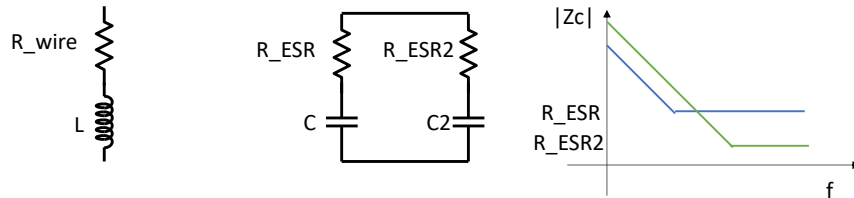
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In this video we're going to look at a few more definitions of quality factor. These are particularly handy when you're thinking about the behavior of individual components.

Real Components Have Losses



- Inductors: winding resistance, magnetization losses in core
- Capacitors: dielectric losses from polarizing dipoles
- Often see parallel capacitors to deal w/ high ESR

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And we like the idea of looking at the quality of individual components because real inductors and capacitors have some loss associated with them. For instance, Inductors are usually made of loops of wire, which has resistance. It's also common to boost inductance by using ferromagnetic materials as inductor cores, and magnetization of materials has some energy loss associated with it. Capacitors have a similar behavior where polarization of the dielectric has a little bit of loss associated with it.

In all of these cases, we model the loss in the component as a series resistor. For capacitors, this is commonly called the equivalent series resistance or ESR.

The main point of this slide is that ESR exists, but while we're talking about ESR, we can explain some puzzling design practices. CLICK In particular, you often see networks of bypass capacitors made of one big capacitor, like a microfarad, in parallel with a much smaller capacitor, like 10nF. If you're just looking at capacitance values, this design choice is puzzling, the smaller capacitor only changes your capacitance by about 1%. However, if you think about ESR, the smaller capacitor can become significant. CLICK This is because big capacitance values tend to come with big ESR values, which means impedance stops dropping with frequency and flattens out at a relatively low corner frequency as pictured in this diagram. CLICK Adding another capacitor in parallel with a lower ESR can extend the effective bandwidth of your capacitor. The overall impedance of the parallel combination

will trace out the bottom of these two curves because the parallel operator is dominated by the smaller impedance.

The Physics Definition of Quality Factor

$$Q = 2\pi \frac{\max(E_{\text{stored}})|^1 \text{ cycle}}{E_{\text{dissipated}}/\text{cycle}}$$

- Handy for remembering definitions of series/parallel RLC Q
- Can be used to define Q for individual components

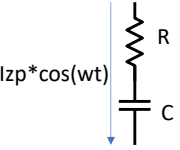
We quantify these losses by going back to the most fundamental definition of quality factor, which is 2π times the maximum of the energy stored during one cycle divided by the energy dissipated in one cycle. This definition means that a system that stores lots of energy without losing much will have a high quality factor. You might note that energy per cycle has units of power, and there are other definitions of Q that rewrite the denominator as an average power.

I like this definition for a few reasons. First, it can help us to remember the Q for series and parallel systems. In series RLC circuits you get a high Q with a small R because current runs back and forth through it every cycle, and a small value will reduce power dissipation. In parallel RLC circuits you get a high Q with a big R because very little current will run in the resistor to be dissipated. This reasoning can help you remember to put R in the numerator or denominator when calculating RLC tank Q values.

Second, I find the definition aesthetically pleasing. It links the behavior of systems to fundamental physical quantities. That's nice.

Third, this definition is useful for defining component Qs. Let's do that.

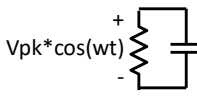
Components have Q that Depends on ω



$$E_{stored} = \frac{1}{2} C V_{C,zp}^2 = \frac{1}{2} C \left(\frac{I_{zp}}{\omega C} \right)^2$$

$$E_{diss} = \frac{I_{pk}^2 R 2\pi}{2 \omega}$$

$$Q = \frac{1}{\omega RC} = \frac{\text{Im}(Z)}{\text{Re}(Z)}$$



$$E_{stored} = \frac{1}{2} C V_{pk}^2$$

$$E_{diss} = \frac{V_{pk}^2 2\pi}{2R \omega}$$

$$Q = \omega RC = \frac{\text{Im}(Y)}{\text{Re}(Y)}$$

- Q for single components is frequency dependent.
- Almost the same derivation for inductors.

We're going to start by imagining that we have a capacitor in series with a resistor. The circuit is being driven by a current source with a zero-to-peak value of I_{zp} . We write down the standard equation for energy stored on a cap, but we run into a problem right away because we don't know the voltage on the capacitor. However, we can find it by multiplying the peak current by the impedance of the capacitor. We're skipping over some subtlety around the fact that the capacitor impedance is imaginary, but we'll circle back to that in greater detail later.

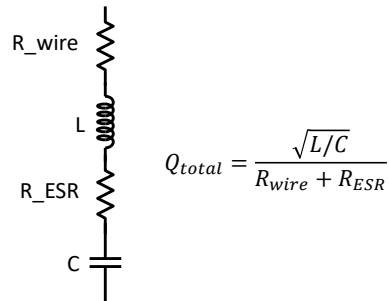
CLICK Finding the power dissipated in a resistor is simpler. We find the average power by averaging $I^2 R$ over one cycle remembering that the average of cosine squared is $\frac{1}{2}$, then we multiply that by one period to find the energy dissipated in a cycle.

CLICK Taking the ratio of these energies and including the factor of 2π from our definition, we find a frequency dependent value for the quality factor of capacitors. That frequency dependence makes some sense because the peak capacitor voltage drops with frequency. Also note that this quantity is the reactance divided by the resistance, which is a fast mnemonic for calculating component Q for series losses in either inductors or capacitors.

CLICK The parallel version of this circuit behaves similarly. We drive it with a voltage source for ease of computation because the voltage will be the same across both branches. Our

final definition of Q is the inverse of the series case, and the mnemonic for parallel circuits is that you take the ratio of susceptance to conductance to find Q.

Find Composite Q with Equivalent Circuits



When you start mixing lossy components together, each of their losses contributes to the overall quality factor of your system, but the summing relationships for those losses depend on whether components are added in series or parallel. I find it easiest to draw an equivalent circuit of the combined elements to calculate an overall Q rather than remembering four separate relationships for series and parallel inductors and capacitors. We'll also see very soon that we can use a mathematical trick called a series-to-parallel transform to make calculating the Q of groups of lossy components easier.

Summary

- Q is defined by energy stored divided by energy dissipated

$$Q = 2\pi \frac{\max(E_{\text{stored}})|^1 \text{ cycle}}{E_{\text{dissipated}}/\text{cycle}}$$

- Components have a ω -dependent quality factor

$$Q_{\text{series}} = \frac{\text{Im}(Z)}{\text{Re}(Z)} \quad Q_{\text{parallel}} = \frac{\text{Im}(Y)}{\text{Re}(Y)}$$

- Individual Q effects combine in bigger circuits.

Series-to-Parallel Transforms

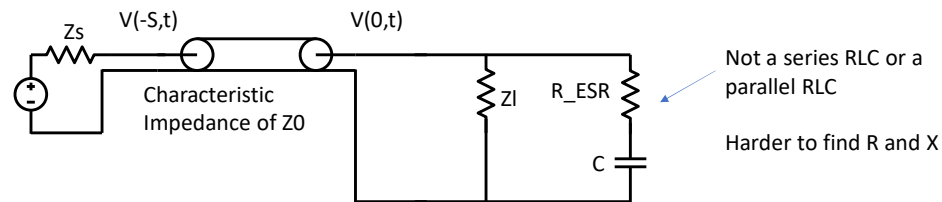
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In this video we're going to examine a funky mathematical construction that lets us calculate the behavior of arbitrary RLC networks at resonance very quickly.

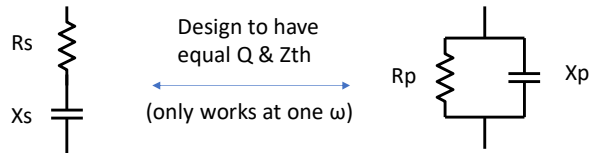
Parasitic R is Inconvenient for Analysis



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I've drawn this circuit to illustrate why we might want to use a series-to-parallel transformation on a circuit. We have already worked out the math describing series and parallel RLC networks, but we can't apply that math to this circuit. Further, that math was kind of a pain that involved a quadratic equation. So having to recalculate formulas for quality factor and natural frequency for every rearrangement of components seems like a headache. However, if we could somehow swap in a model of the capacitor that had the ESR in parallel with the cap instead of in series, this becomes a simple parallel RLC circuit.

Fix: Make Parallel Circuit w/ Same Q at one ω



$$Z_{th} = jX_s + R_s = R_p || jX_p = \frac{jX_p R_p}{R_p + jX_p} = \frac{jX_p R_p^2 + X_p^2 R_p}{R_p^2 + X_p^2}$$

Equate $Z_s = Z_p$ and clean up imaginary terms

$$Q = \frac{X_s}{R_s} = \frac{R_p}{X_p}$$

Equate component Q

$$X_s = X_p \frac{R_p^2}{R_p^2 + X_p^2} = X_p \frac{Q^2}{Q^2 + 1}$$

Equate imaginary parts ~ equal for high Q

$$R_s = R_p \frac{X_p^2}{R_p^2 + X_p^2} = R_p \frac{1}{Q^2 + 1}$$

Equate real parts ~ scale by Q^2 for high Q

We can make a parallel model of a component that matches the series behavior at one frequency by making sure the impedance and quality factors of the series and parallel models are equal. This only works at one frequency because the series and parallel formulas are quite different from one another, but it's OK because we often only really care about the behavior of RLC systems at resonance. Also, note that though I've drawn some schematics using capacitors, we're going to do our math with a generic reactance to make sure our results apply to inductors too.

I've started computing the transformation by equating the impedance of the series combination with the parallel combination here. We're going to need to split this into real and imaginary parts soon, so CLICK we have to do some algebra to rationalize our parallel expression. CLICK the second step in this process is equating the component quality factors for the series and parallel models. Finally, we can split up the impedance into real and imaginary parts. CLICK The imaginary part gives us a relation for the series reactance in terms of the parallel reactance and Q. This expression is interesting when Q is high: our transform says we don't need to change the value of reactance for big values of Q. CLICK The real part gives us a relation between the series resistance and the parallel resistance in terms of Q. For big values of Q we see that the series and parallel resistances are related by Q^2 .

This resistance equation can make resistors look substantially larger or smaller, which captures the underlying physics that current in capacitive and inductive branches of a parallel RLC circuit is amplified by Q compared to the resistive branch. This is a useful impedance transformation that we can achieve just by putting a load resistance in series with an inductor or capacitor. We'll start using it very soon.

Summary

- You can equate series and parallel RC or RL networks at one frequency, which is convenient for analysis.
- Do so by setting Z and Q equal in series and parallel networks
- Equations capture a resistance transform

$$X_p = X_s \frac{Q^2 + 1}{Q^2} \qquad R_p = R_s(Q^2 + 1)$$