Lecture 19:
Linearity and Distortion 1

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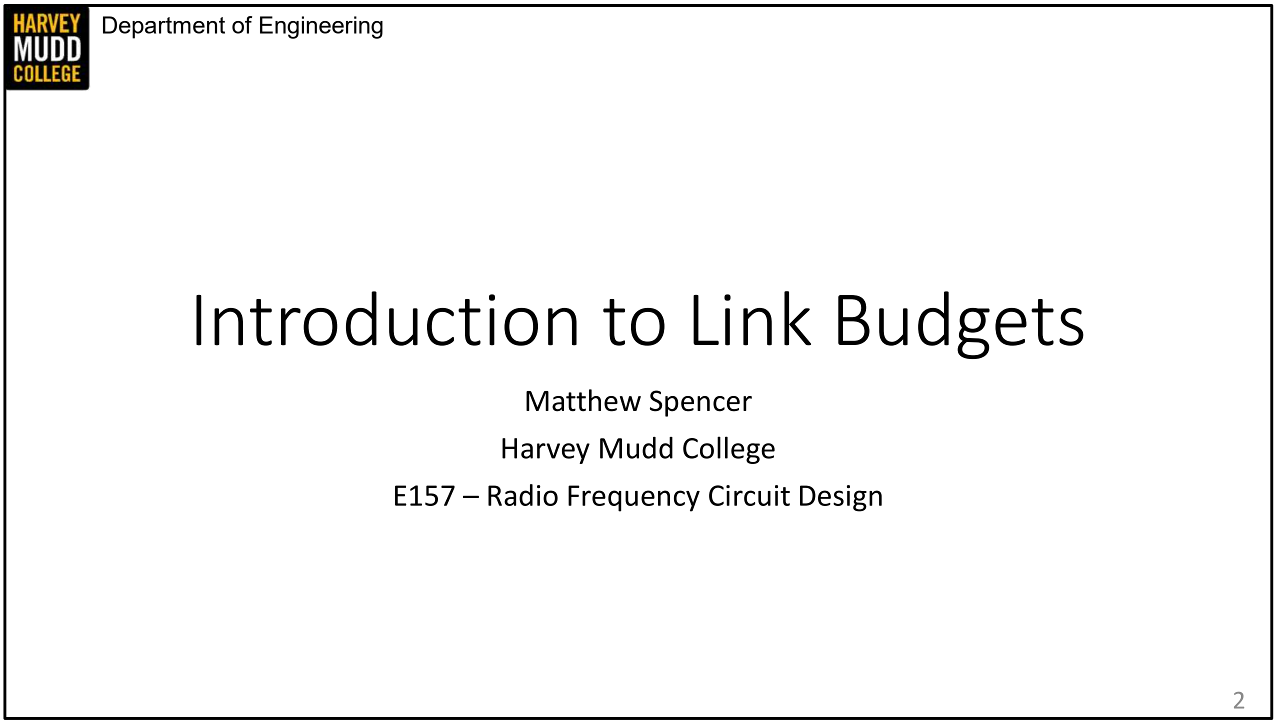

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Introduction to Link Budgets

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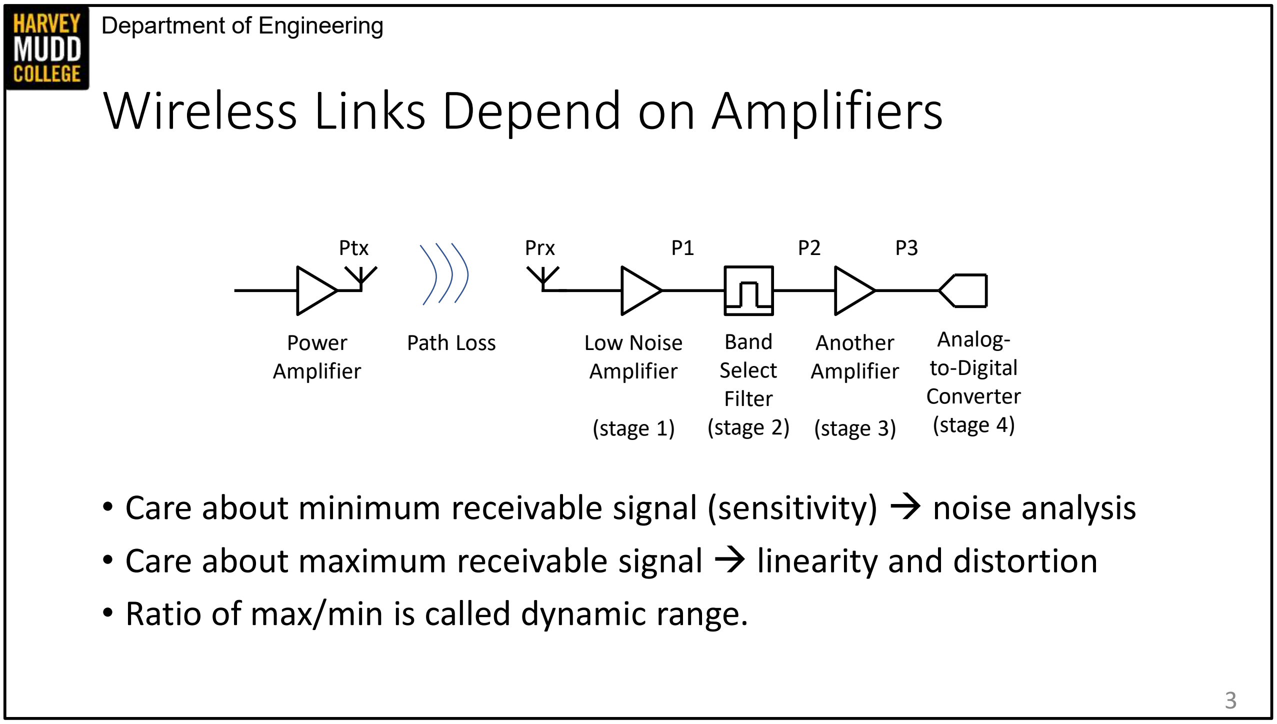
In this video we’re going to discuss how we’re going to think about communication links for the rest of the semester.

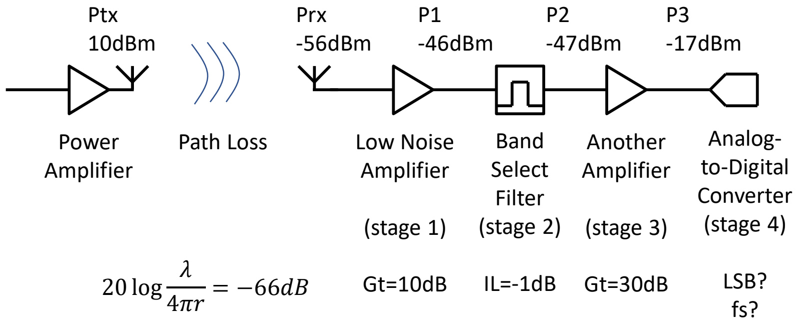


Wireless Links Depend on Amplifiers

- Care about minimum receivable signal (sensitivity) → noise analysis
- Care about maximum receivable signal → linearity and distortion
- Ratio of max/min is called dynamic range.

A link budget is an accounting of power levels at different parts of a receive chain, so for the receiver pictured here, it would be a spreadsheet that tracks the values of each of Ptx, Prx, P1, P2 and P3. Keeping this document lets you make important design decisions, like how many amplifiers to use in the receive chain. It also lets you predict important performance parameters of your system. One crucial example is sensitivity, which measures the smallest signal that you can receive. Finding the sensitivity requires us to analyze the electrical noise in our system, so we'll be developing a theory how much noise your system generates soon. Another important parameter is the maximum power our system can tolerate, and finding that maximum power requires us to generate a different theory about non-linear behavior in amplifiers. These two design parameters define a maximum and a minimum acceptable signal, and the ratio of those maximum and minimum signals is called dynamic range. Dynamic range is the king of specifications for many analog systems, and we'll care about it in this class too.

eg: Transmit 10dBm, 3Grad/s, 10m Away



Ptx 10dBm

Prx -56dBm

P1 -46dBm

P2 -47dBm

P3 -17dBm

Power Amplifier

Path Loss

Low Noise Amplifier (stage 1)

Band Select Filter (stage 2)

Another Amplifier (stage 3)

Analog-to-Digital Converter (stage 4)

$20\log\frac{\lambda}{4\pi r} = -66dB$

Gt=10dB

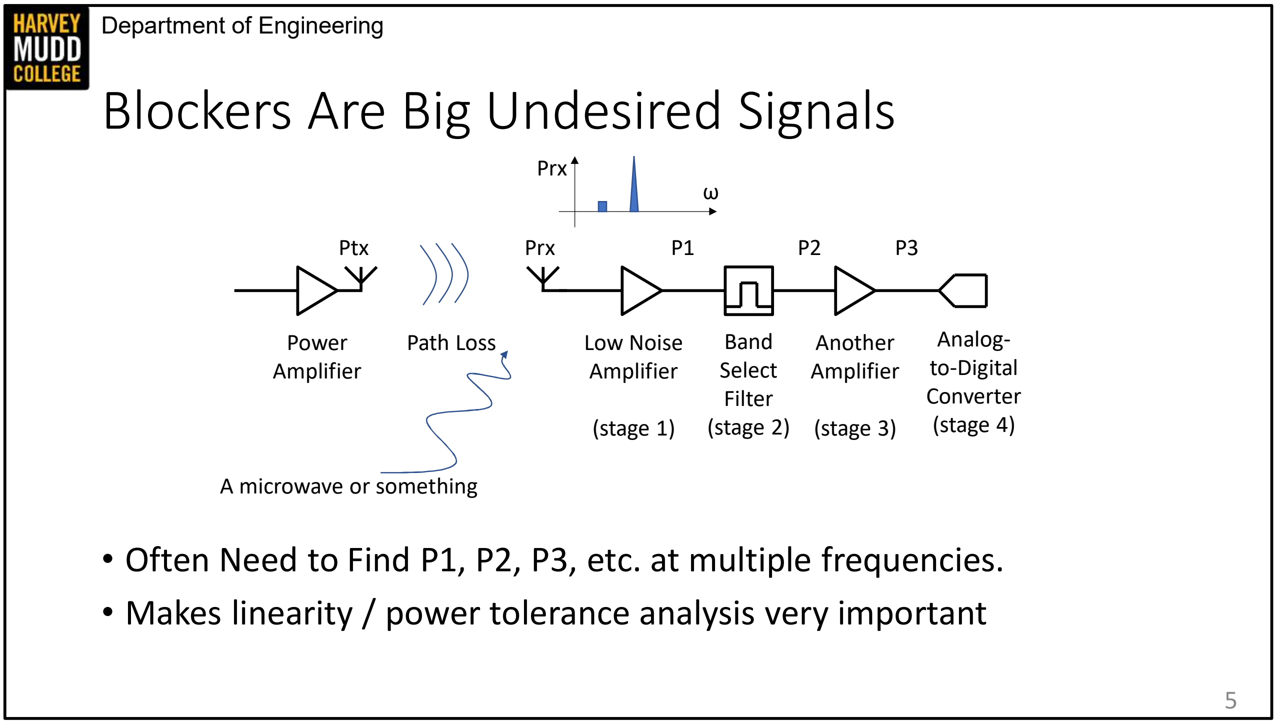
IL=-1dB

Gt=30dB

LSB?
fs?

- This example ignores important filters, TX/RX switches and mixers
- Need to add more details about linearity and noise at P1, P2, P3

So here’s a simple example. We assume we’re transmitting a 10dBm signal at 3Grad/s from an antenna that is 10m away from our receiver. We’ve noted that our transmit power is 10dBm on the left. The signal goes through some path loss on the way to the receiver (and note that we’re ignoring transmit and receive antenna gain in this example), so we calculate the path loss as -66dB. That means the receive antenna picks up -56dBm of power. There’s a 10dB gain on our first amplifier stage, which means P1 is -46dBm, then the insertion loss of a band pass filter reduces our power to -47dBm, and another amplifier gives us a final power level of -17dBm at the analog to digital converter. That value is important to know: it lets us answer questions about the ADC like how small should the least significant bit be, and what should the sampling frequency be. Even with this simplified model that ignores components like filters, TX/RX switches, and mixers, and that ignores noise and linearity, we can get some interesting information. Ad we add details to this model over the next few weeks, we’ll get more and more interesting information.



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Blockers Are Big Undesired Signals

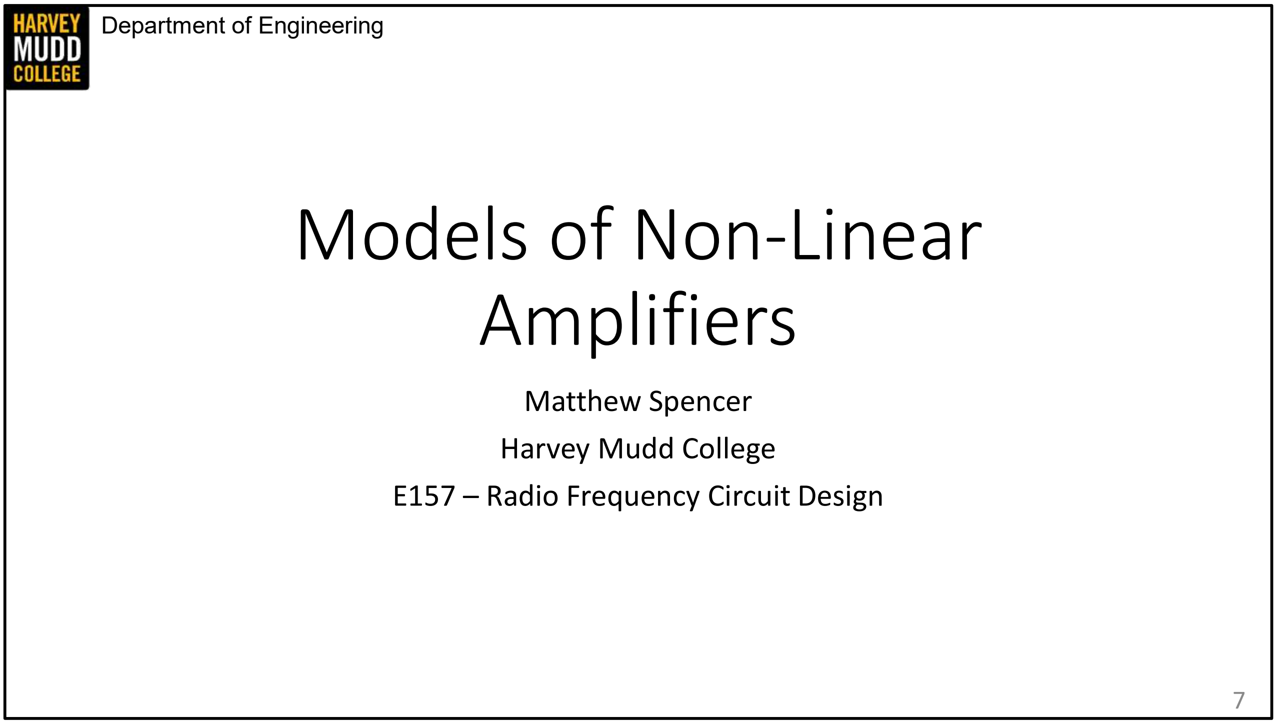
- Often Need to Find P1, P2, P3, etc. at multiple frequencies.
- Makes linearity / power tolerance analysis very important

When we’re making link budgets we also have to think about undesired signals. This image depicts a type of undesired signal called a blocker, which is a very high power signal that has a frequency near to our desired signal. The classic example is if you’re on wifi and someone turns on a poorly designed microwave: both wifi and microwaves live in the crowded 2.4GHz unlicensed industrial, scientific and medical band, which is abbreviated ISM, but microwaves throw out kilowatts of power while wifi signals can be quite weak. I’ve sketched an example of a spectrum with a blocker above the receive chain.

A blocker like the microwave can overload our amplifiers, which causes the whole receive chain to fail. That means when we make link budgets we often need to make them at multiple frequencies to assess how the combination of filters and amplifiers that we picked will work for blocking signals in addition to our frequency of interest. It also means that the linearity and distortion behavior of a receiver can be just as important as the noise behavior, especially in congested wireless environments.

Summary

- A link budget is an accounting of signal power, noise power and distortion power at different parts of a signal chain.
- Blockers add complexity to a link budget, requiring more frequencies to be considered.



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Models of Non-Linear Amplifiers

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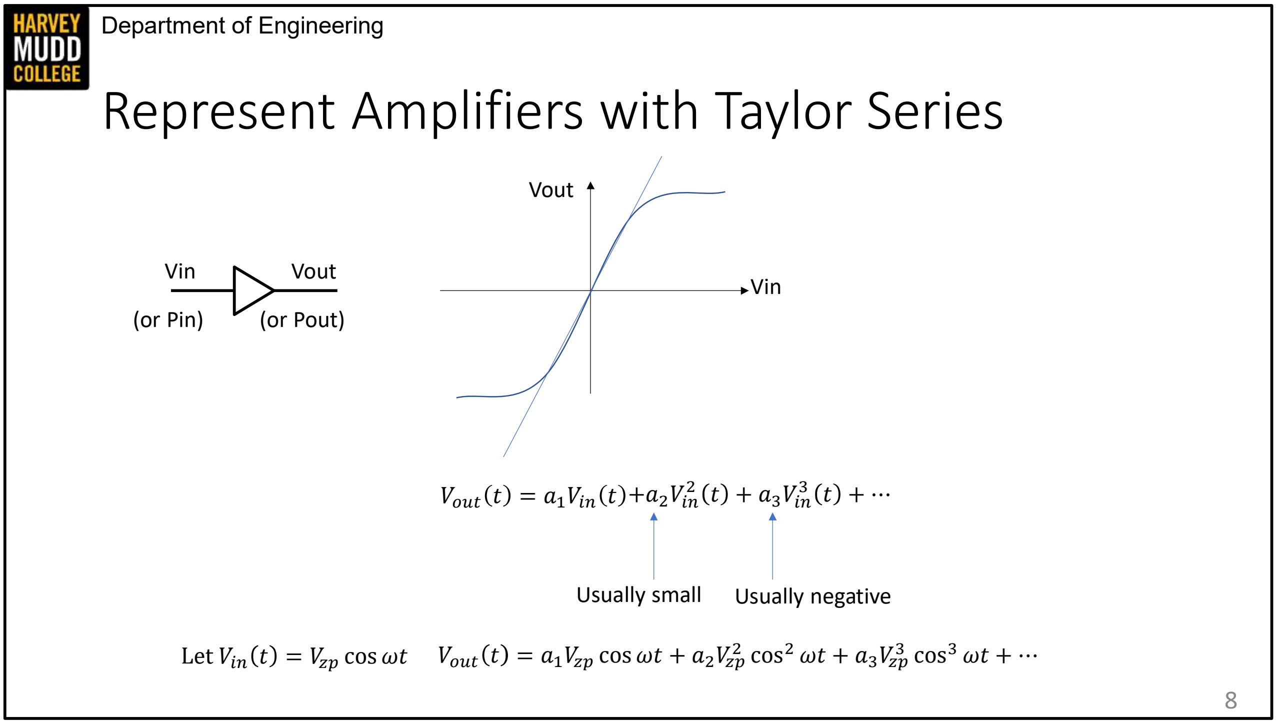
In this video we're going to start analyzing the maximum signal handling capability of receiver chains by building a model of a non-linear system. This is an interesting departure from the rest of the course, where everything we analyzed was very linear. Though this video has the word amplifier in the title, the techniques we're going to learn apply to other non-linear devices, like power detectors, too.



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Represent Amplifiers with Taylor Series

$$V_{out}(t) = a_1 V_{in}(t) + a_2 V_{in}^2(t) + a_3 V_{in}^3(t) + \cdots$$

Usually small

Usually negative

Let $V_{in}(t) = V_{zp} \cos \omega t$ $\quad V_{out}(t) = a_1 V_{zp} \cos \omega t + a_2 V_{zp}^2 \cos^2 \omega t + a_3 V_{zp}^3 \cos^3 \omega t + \cdots$

8

This page shows a schematic of an amplifier with an input voltage and an output voltage. The schematic also reminds us that we can think of the amplifier as having an input power and an output power, which ties back to the concepts of power gain that we talked about earlier. You can calculate the amplifier's input and output power by squaring the voltage and dividing by the impedance, which is usually Z0 in an RF system.

The slide also shows an equation and a graph that indicate the relation between Vout and Vin, which is called the transfer characteristic. That transfer characteristic is pretty simple for an ideal amplifier, it just makes its output linearly bigger than its input, so we see the equation multiplies the input by the voltage gain a1, and the graph is just a line with a slope of a1.

CLICK Unfortunately, real amplifiers aren't perfectly linear. At minimum, they're limited by their power supplies, and there could be other non-linearities in the amplifier too. That means we need to represent amplifiers with Taylor series that capture their non-linear behavior.

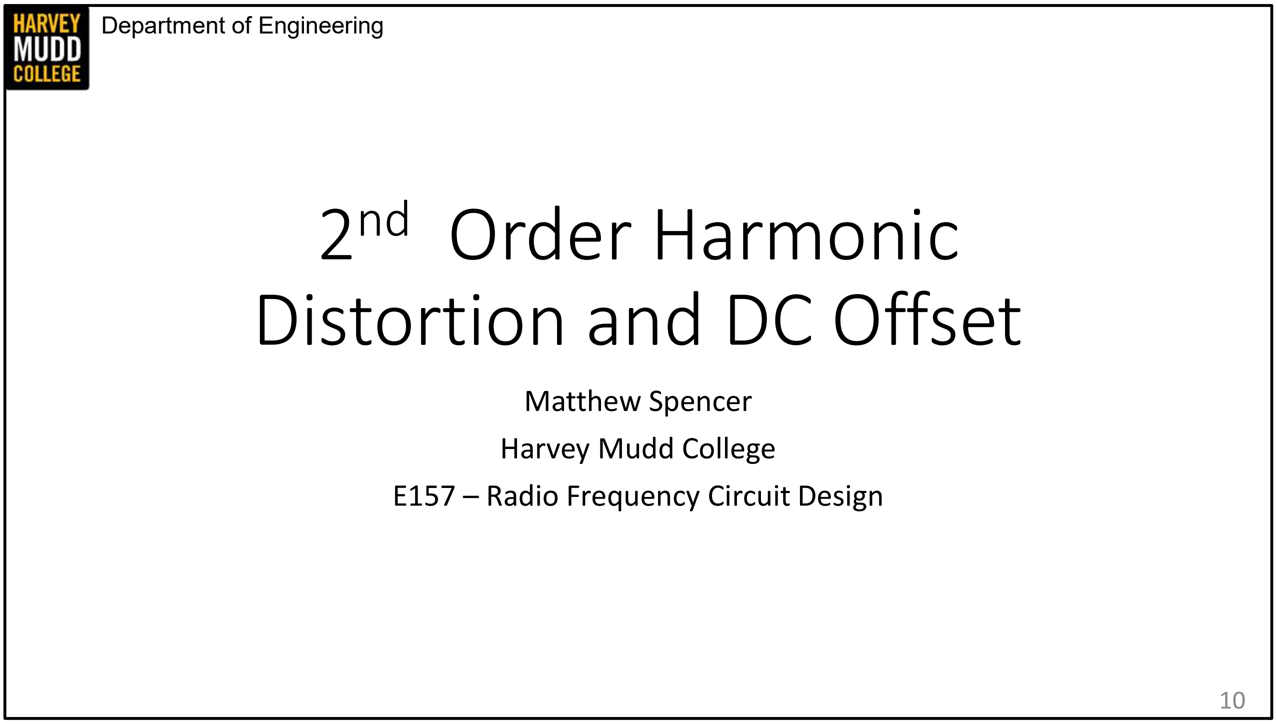
CLICK We can make some guesses about the value of the Taylor coefficients for normal amplifiers. Amplifier curves are usually odd, or close to it, because they try so hard to be linear. That means that the a2 coefficient is usually small. Also, amplifiers are usually compressive because they clip on the power supply, so the a3 coefficient is usually negative, which starts cancelling out the straight line from the a1 coefficient at high Vin values.

CLICK Finally, when we're working with transfer functions its easy to forget that the inputs to an amplifier are usually time varying. In this last set of equations, I've assumed that our Vin is a sinusoid, which is normal for RF systems, to remind us that Vout is a time-varying quantity. In the next few videos, we're going to see that squaring and cubing sinusoids has surprising effects on our output spectrum.

The transfer function captures an instantaneous, dynamics-free picture of how an amplifier works.

Summary

- We represent non-linear amplifiers with Taylor Series
- The 2nd order coefficient of Taylor is small because amplifiers have odd transfer characteristics
- The 3rd order coefficient of Taylor is negative because amplifiers are compressive at large signal levels



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2^{nd} Order Harmonic Distortion and DC Offset

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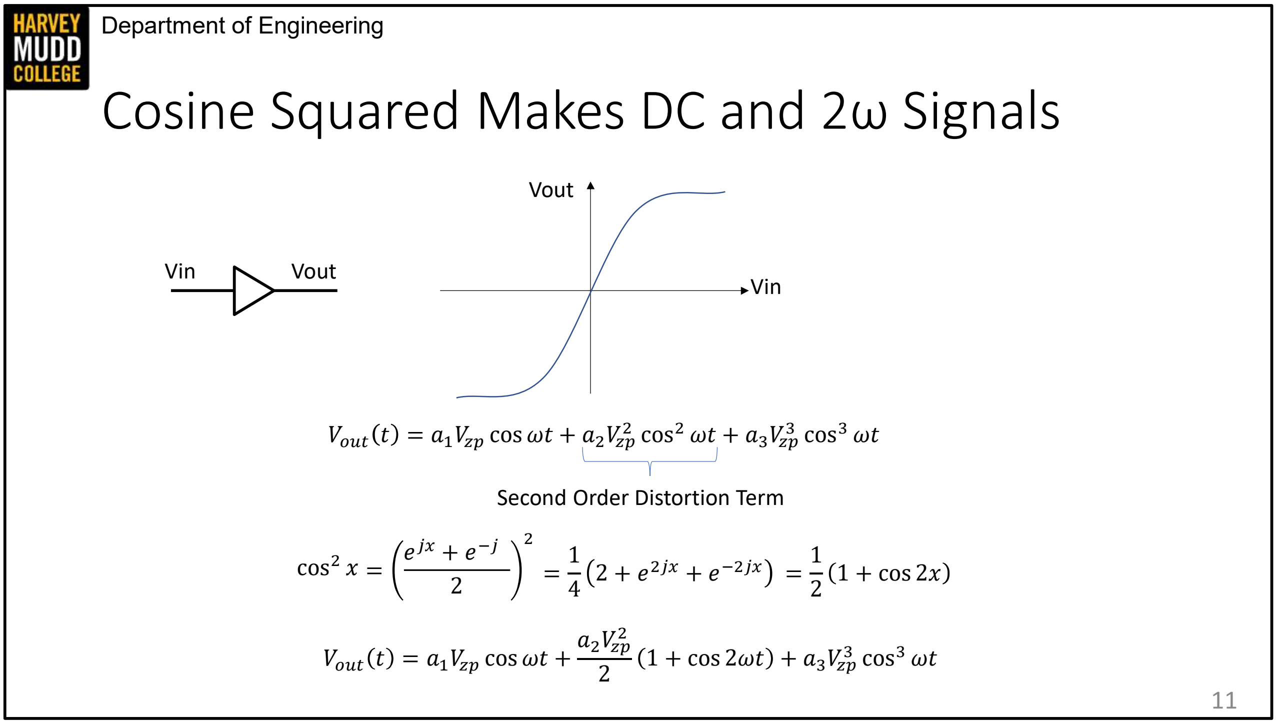
In this video we’re going to start examining the output of a nonlinear amplifier driven by a cosine more closely.



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Cosine Squared Makes DC and 2ω Signals

$$V_{out}(t) = a_1 V_{zp} \cos \omega t + \underbrace{a_2 V_{zp}^2 \cos^2 \omega t}_{\text{Second Order Distortion Term}} + a_3 V_{zp}^3 \cos^3 \omega t$$

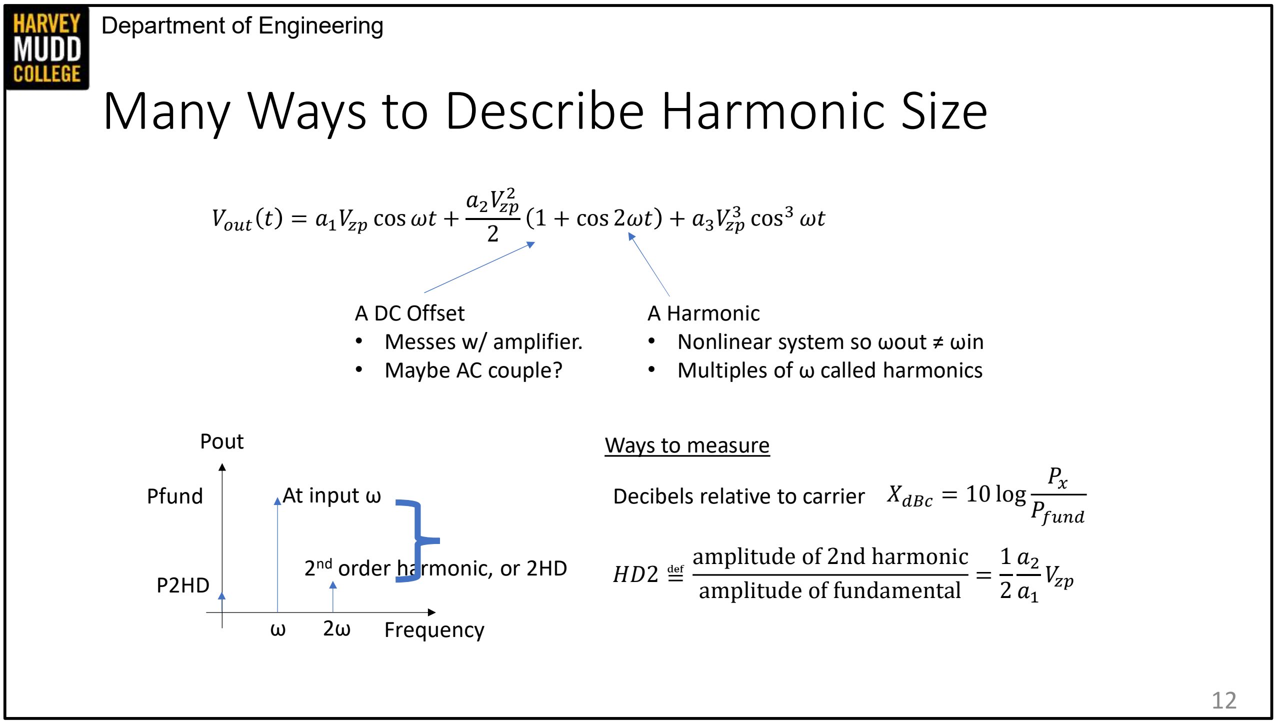
$$\cos^2 x = \left(\frac{e^{jx} + e^{-j}}{2}\right)^2 = \frac{1}{4}\left(2 + e^{2jx} + e^{-2jx}\right) = \frac{1}{2}(1 + \cos 2x)$$

$$V_{out}(t) = a_1 V_{zp} \cos \omega t + \frac{a_2 V_{zp}^2}{2}(1 + \cos 2\omega t) + a_3 V_{zp}^3 \cos^3 \omega t$$

11

We’re going to do that by applying some trigonometry to the squared term in the amplifier’s behavior, which is referred to as the second order distortion term.

CLICK We need to simplify this cosine squared term, but no one remembers trig identities, so we’re going to use Euler’s Formula to think about complex exponentials instead.
CLICK After we do that, it’s pretty easy to square e^jx plus e^-jx, which leaves us with a polynomial that contains e^2jx and e^-2jx,
CLICK Those two complex exponentials can be combined back into a cosine in the final step of our derivation, which leaves us with a simple expression that doesn’t involve higher powers of cosines.
CLICK Finally, we substitute our cosine identity back into our Vout(t) expression



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Many Ways to Describe Harmonic Size

$$V_{out}(t) = a_1 V_{zp} \cos \omega t + \frac{a_2 V_{zp}^2}{2}(1 + \cos 2\omega t) + a_3 V_{zp}^3 \cos^3 \omega t$$

A DC Offset
- Messes w/ amplifier.
- Maybe AC couple?

A Harmonic
- Nonlinear system so ωout ≠ ωin
- Multiples of ω called harmonics

Ways to measure

Decibels relative to carrier $X_{dBc} = 10 \log \frac{P_x}{P_{fund}}$

$$HD2 \overset{\text{def}}{=} \frac{\text{amplitude of 2nd harmonic}}{\text{amplitude of fundamental}} = \frac{1}{2}\frac{a_2}{a_1} V_{zp}$$

This equation has some terms that are a bit surprising.
CLICK first, the cosine squared creates a DC term. So putting a high frequency wave into our second order harmonic distortion causes a DC shift in the output of our amplifier. This can be a big problem for some amplifiers because it messes with their operation. However, it's common to fix this behavior by putting capacitors in series with amplifiers, a practice called AC coupling.
CLICK the second term in this equation creates a frequency component at twice the input frequency, which is referred to as the second harmonic. Harmonics are signals that have frequencies that are multiples of some lower frequency, which is called a fundamental. This is very different from linear systems, which always have output sinusoids at the same frequency as input frequency.
CLICK We can make a quick picture of this, which shows that our amplifier's output spectrum has the expected linear gain at the input frequency omega, another component at two omega, and a small DC offset. This plot is in units of power, which is a common way to draw spectra for RF systems where we're often more concerned with power gain than voltage gain. As a reminder, we can convert from our zero-to-peak amplitude to power by squaring the voltage amplitude and dividing by twice the resistance it sees, which is probably Z0 if it is matched to the surrounding RF system.
CLICK This graph is a great summary of what's going on, but we can come up with some concise numerical summaries too. Frist, it's common to describe the power of the second

harmonic in relation to the power of the fundamental signal. We do that by using yet another funny decibel unit called dBc, or decibels relative to the carrier. We'll talk about where the term carrier comes from in a later video. For now just remember that it refers to the fundamental signal. A quantity in dBc is ten times the log of the thing we're measuring divided by the power of the fundamental

CLICK so when we say that the second harmonic is some number of dBc below the fundamental frequency, we're effectively measuring this vertical height in decibels.

CLICK finally, HD2 is a specific measure of the size of second harmonic relative to the input. It's the voltage amplitude of the second harmonic divided by the fundamental frequency, and the expression you get when dividing those two coefficients is given here. This ratio gets linearly larger as Vzp increases.

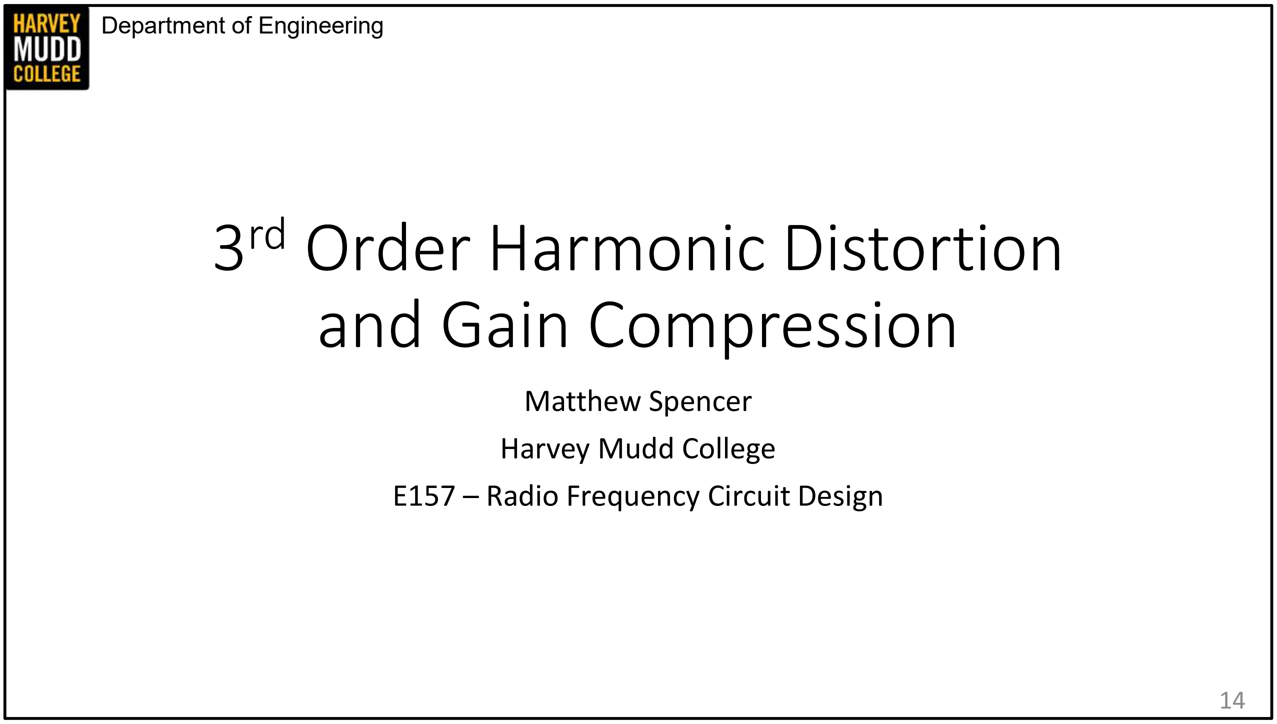
Summary

- Second order distortion term creates DC offset and a harmonic at 2ω
- DC offset can be problematic in certain designs, but often AC coupled
- Harmonics can be measured in decibels relative to carrier

$$X_{dBc} = 10 \log \frac{P_x}{P_{fund}}$$

- Second harmonic can be measured using the HD2 quantity

$$HD2 \stackrel{\text{def}}{=} \frac{\text{amplitude of 2nd harmonic}}{\text{amplitude of fundamental}} = \frac{1}{2}\frac{a_2}{a_1}V_{zp}$$



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3^{rd} Order Harmonic Distortion and Gain Compression

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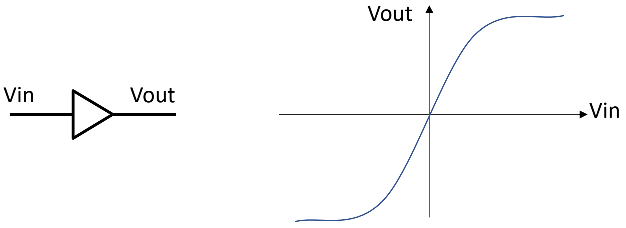
In this video we’re going to continue exploring the behavior of non-linear amplifiers

Cosine Cubed Makes 3ω Signal and Original ω



$$V_{out}(t) = a_1V_{zp}\cos\omega t + a_2V_{zp}^2\cos^2\omega t + \underbrace{a_3V_{zp}^3\cos^3\omega t}_{\text{Third Order Distortion Term}}$$

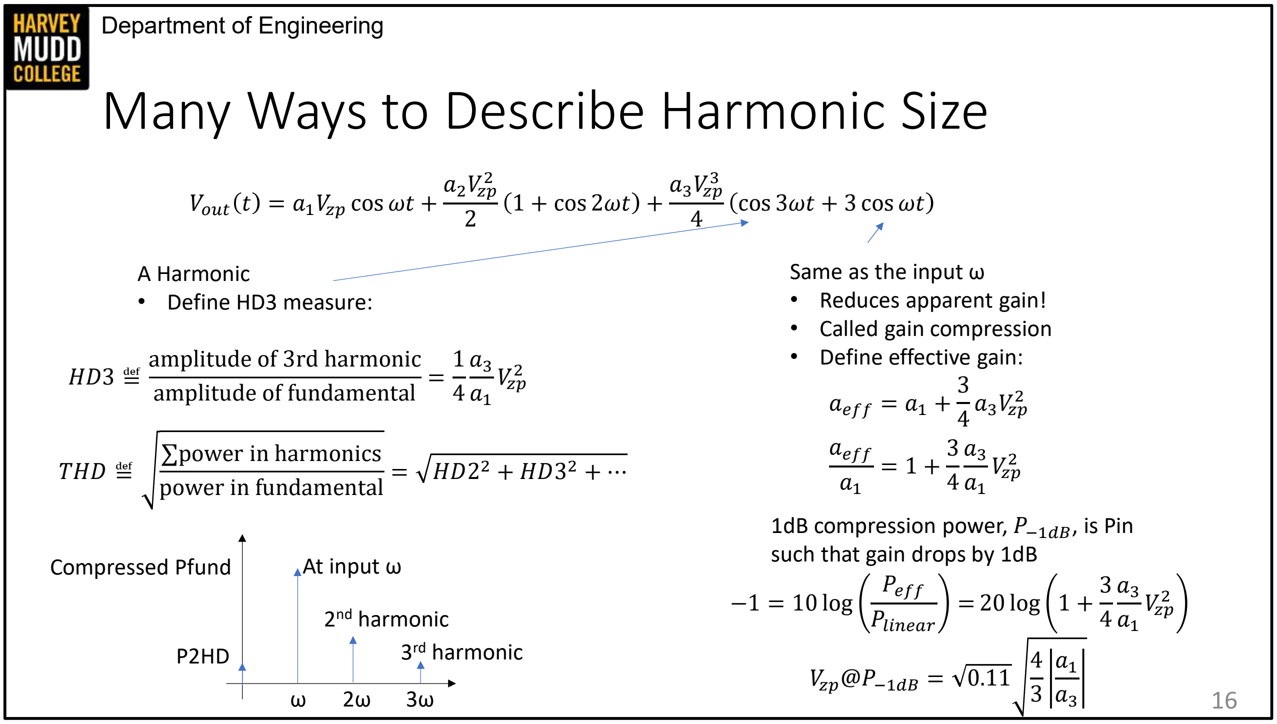
$$\cos^3 x = \left(\frac{e^{jx}+e^{-jx}}{2}\right)^3 = \frac{1}{8}\left(e^{3jx}+e^{-3jx}+3e^{jx}+3e^{-j}\ \right) = \frac{1}{4}(\cos 3x + 3\cos x)$$

$$V_{out}(t) = a_1V_{zp}\cos\omega t + \frac{a_2V_{zp}^2}{2}(1+\cos 2\omega t) + \frac{a_3V_{zp}^3}{4}(\cos 3\omega t + 3\cos\omega t)$$

And we’re going to do so by looking at the cubic term in our equation, which is referred to as the third order distortion term.

CLICK Like before, we’re going to derive a trig identity that we have probably all forgotten by making a polynomial out of Euler’s formula. It’s worth noting that I calculate the coefficients for this polynomial using Pascal’s triangle, so a cubic polynomial has coefficients 1, 3, 3 and 1. The result of this process is on the right: cosine cubed of x is equal to one quarter of cosine 3x plus 3 cosine x.

CLICK And also like before we can substitute that identity into our Vout equation.



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$$V_{out}(t) = a_1 V_{zp} \cos \omega t + \frac{a_2 V_{zp}^2}{2}(1 + \cos 2\omega t) + \frac{a_3 V_{zp}^3}{4}(\cos 3\omega t + 3 \cos \omega t)$$

A Harmonic
- Define HD3 measure:

$$HD3 \stackrel{\text{def}}{=} \frac{\text{amplitude of 3rd harmonic}}{\text{amplitude of fundamental}} = \frac{1}{4}\frac{a_3}{a_1}V_{zp}^2$$

$$THD \stackrel{\text{def}}{=} \sqrt{\frac{\sum \text{power in harmonics}}{\text{power in fundamental}}} = \sqrt{HD2^2 + HD3^2 + \cdots}$$

Same as the input ω
- Reduces apparent gain!
- Called gain compression
- Define effective gain:

$$a_{eff} = a_1 + \frac{3}{4}a_3 V_{zp}^2$$

$$\frac{a_{eff}}{a_1} = 1 + \frac{3}{4}\frac{a_3}{a_1}V_{zp}^2$$

1dB compression power, P_{-1dB}, is Pin such that gain drops by 1dB

$$-1 = 10 \log\left(\frac{P_{eff}}{P_{linear}}\right) = 20 \log\left(1 + \frac{3}{4}\frac{a_3}{a_1}V_{zp}^2\right)$$

$$V_{zp}@P_{-1dB} = \sqrt{0.11}\sqrt{\frac{4}{3}\left|\frac{a_1}{a_3}\right|}$$

16

Just like the second order distortion, third order distortion has some surprising properties.

CLICK First, we see that the third order term produces a harmonic at 3 omega. Like before, we can define an HD3 measure which is the voltage amplitude of the 3rd harmonic divided by that of the fundamental. The ratio of the coefficients of those terms comes out to one quarter a3 over a1 times Vzp^2. One interesting note is that HD2 grew linearly with Vzp, while HD3 grows as Vzp squared, which shows that higher harmonics become more significant as you provide more and more power.

It's useful at this point to define a measure called total harmonic distortion, or THD, which measures how distorted a sine wave is. THD is the sum of power in harmonics divided by the power in the fundamental. We can compute it adding our harmonic distortion measures in quadrature, which works because each measure is a ratio of a harmonic voltage over a fundamental voltage and you can square ratios of voltages to get ratios of powers. (THD is useful because … quantify by application)

CLICK The other term in our third order distortion expression is 3cos omega t, which falls directly on top of our input. Because a3 is often negative, this term has the effect of reducing the gain of the system, which is a way of indicating that the compressive behavior of the amplifier has started. Accordingly, this effect is called gain compression. We can describe gain compression mathematically by gathering terms that are multiplied by cosine of omega t to define the effective gain, then dividing through by a1 to find the ratio of effective gain to linear gain.

CLICK This ratio of effective gain to linear gain is useful for defining an easy to measure parameter called P-1dB. P-1dB is the input power that causes our gain to be compressed by 1dB. We find it by taking 20 log of our effective gain, using the 20 as our coefficient instead of 10 because we're measuring power instead of voltage, and setting that equal to -1. We can flip the equation around to find the value of Vzp that would cause us to reach P-1dB.

CLICK Finally, here's a plot showing the frequency contant that we've introduced so far. Second and third order harmonics, gain compression and a DC offset.

Summary

- 3$^{\text{rd}}$ order distortion term creates gain compression and 3ω harmonic
- 3$^{\text{rd}}$ harmonic described by HD3 measure

$$HD3 \overset{\text{def}}{=} \frac{\text{amplitude of 3rd harmonic}}{\text{amplitude of fundamental}} = \frac{1}{4}\frac{a_3}{a_1}V_{zp}^2$$

- Effect of all harmonics combined is measured by THD

$$THD \overset{\text{def}}{=} \sqrt{\frac{\sum \text{power in harmonics}}{\text{power in fundamental}}} = \sqrt{HD2^2 + HD3^2 + \cdots}$$

- Gain compression measured by P_{-1dB} $\quad V_{zp}@P_{-1dB} = \sqrt{0.11}\sqrt{\frac{4}{3}\left|\frac{a_1}{a_3}\right|}$