Lecture 04: Sinusoidal Inputs

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Sine Waves on Infinite Lossless Transmission Lines

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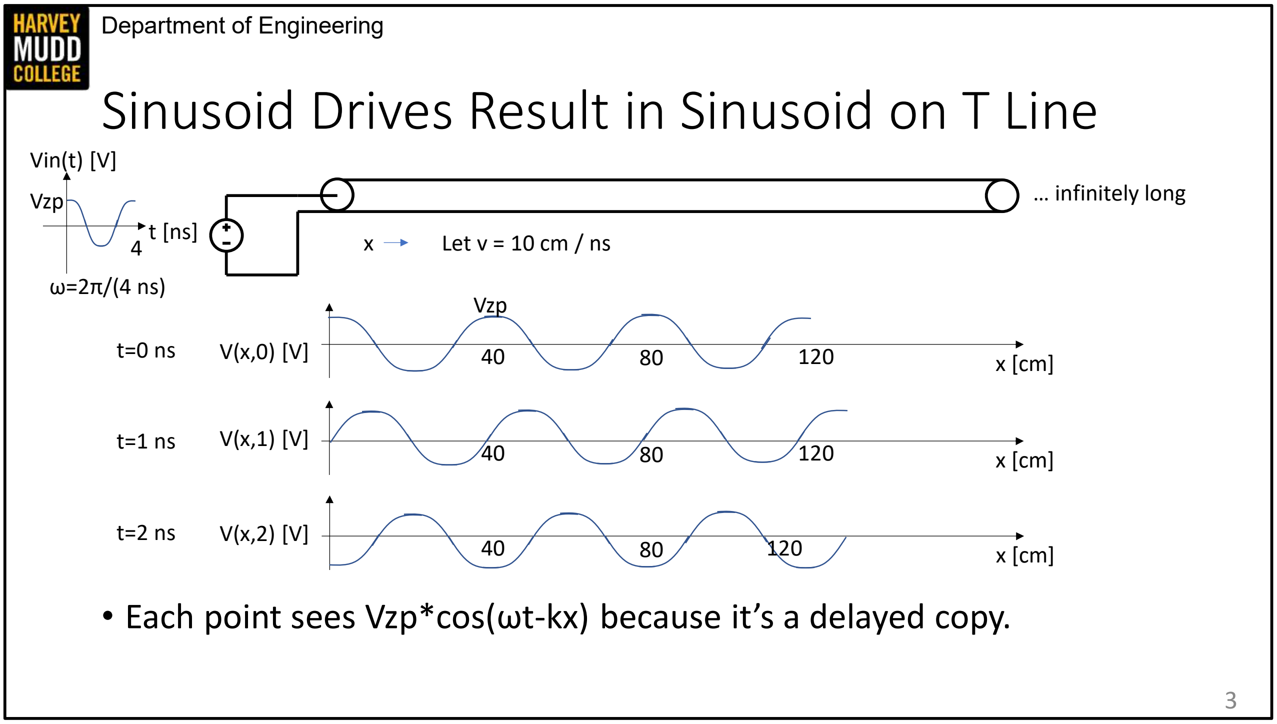
2

In this video we're going to examine what transmission line voltages look like in sinusoidal steady state. This is important because we know by Fourier that we can represent any wave as a sum of sinusoids, which means that a good understanding of sinusoidal steady state will let you understand every wave you run into.



I've drawn a circuit that could result in sinusoidal steady state voltages on a transmission line at the top of this page. You can see Vin(t) is a cosine wave here, and we'll make a mathematical formulation of it on the next page. The wave has a period of 4ns, which is that same thing as saying it picks up two pi of phase in 4 ns, and that's how we calculate the angular frequency. We're also saying that Vin(t) has a zero-to-peak amplitude of Vzp. I've drawn V(x,t) for a few different values of t to show how this Vin(t) propagates down the transmission line.

We can see at time zero that the Vin(t) sinusoid has propagated everywhere on this infinite line. The fact that the sine wave has already propagated an infinite distance at time zero might trouble you, but this is just a standard consequence of sinusoidal steady state analysis. We assume the sinusoidal drive has been going on forever when we do this kind of math. The x dimension of the sie wave has been stretched by the velocity of the transmission line as usual, you can see that stretching by looking at the first peak of the 0ns curve, it's at 40cm, which is 10 cm per ns times the 4ns period. The amplitude of the wave is Vzp everywhere, the same as Vin(t). That's because we have a source impedance of zero in this example, so the divider between the source impedance and the driving point impedance has a magnitude of 1: Z0 over (Zs+Z0) is one if Zs=0. In a general case, the zero-to-peak voltage on the line will be set by the source divider.

There are a few different ways to look at the time evolution of these voltage distributions. If you look at the peaks of the distribution, you can see that waves are propagating to the right at 1ns and 2ns. They propagate by 10cm/ns, in keeping with velocity, which also has the effect of shifting them by pi/2 radians per ns. Another way to look at these is to look at a single point in space, the x=40cm point for example, where you'll see that each point has a sinusoid of frequency omega living on it. This makes sense because every point in the transmission line is a delayed version of Vin(t). Finally, if you freeze time and move back and forth in x, you'll also see a sinusoid. The same argument applies: each point in x adds a different amount of delay to a sinusoid that has been going forever, so we expect to see a different phase at each x value.

We can capture the periodicity in t and the periodicity in x with the equation I've written at the bottom of the slide. Each point has cos(wt) running on it, shifted right in phase by some amount that depends linearly on distance. This raises the question: what's the value of k?

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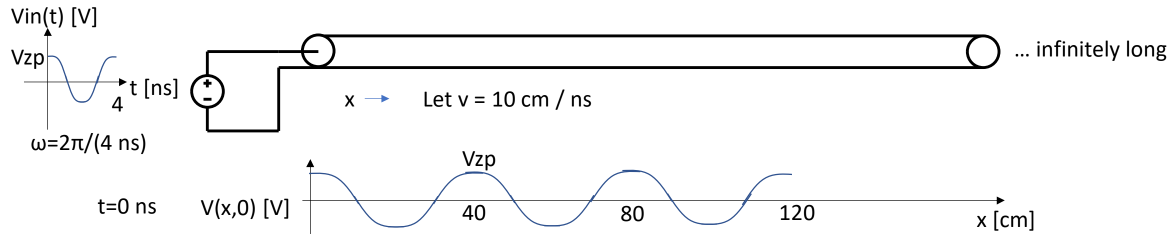
Wave Number is "Spatial Frequency"



$V_{in}(t) = V_{zp}\cos(\omega t)$ Formulation for Vin(t)

$V(x,t) = V_{zp}\cos(\omega t - kx)$ Describes V(x,t), solution to wave equation

$k = \frac{\omega}{v} = \frac{2\pi}{\lambda}$ Proofs: sub into wave equation, pick up 2π each λ, distance per radian

- Wavenumber k has units [rad/m], analogous to ω having units [rad/s]

4

To find out, we need to turn our pictures of sine waves into equations. I've copied over the t=0ns picture for reference.

CLICK We've already got an easy entry point, which is rewriting Vin(t) as Vzp times cosine of omega t, and CLICK we also wrote an expression for V(x,t) on the previous page. It's Vzp times cosine of omega t minus k x. It's worth noticing that this is a solution to the wave equation because it's just a horizontally stretched version of f(x-vt).

CLICK finally, we need to figure out what the constant k is in this equation. I've written the answer up here and listed a few justifications. The first justification is that we need our solution to solve the wave equation, and the only way for it to do so is for k to be equal to omega over v. Talking through the derivatives in our head: two x derivatives will give k^2 on the left side of the wave equation and two t derivatives will leave omega^2/v^2 on the right side, so k needs to be omega over v. Second, we can argue that each wavelength of the wave on the line has to result in an additional two pi of phase being accrued in the argument of the cosine function, so that we're back at the same point on the wave each period. The easiest way to write that is that k equals 2pi over lambda meters so that if x were equal to lambda, kx would be two pi radians. The third way to think about k is to imagine it's telling you the distance per radian on the transmission line, and you can imagine that omega over v is calculating that quantity because omega has units of radians per time and v has units of distance per time. Finally, as a sanity check, if you substitute v=lambda*f into the 2pi over lamba relation, you come back to the omega over v relation.

The upshot of all of this is that we have defined something like a spatial frequency. k has units of radians per meter, which is analogous to omega having units of radians per second. If t isn’t changing, k tells us how far we have to travel on a transmission line before our wave repeats.

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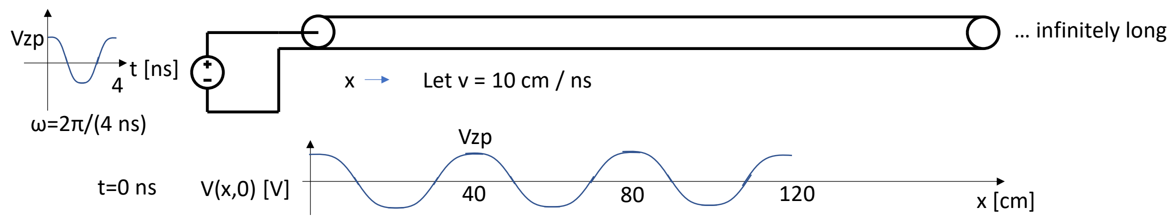
Complex Exponentials Make Math Easier



Real World	$V_{in}(t) = V_{zp}\cos(\omega t)$ $V(x,t) = V_{zp}\cos(\omega t - kx)$	Sub in e^jω → ← Take the real part	$V_{in}(t) = V_{zp}e^{j\omega t}$ $V(x,t) = V_{zp}e^{j(\omega t - kx)}$	Analytic Representation

- Math with complex exponentials is easier. Represents ϕ relations.

5

There’s one further complication with how we represent sinusoids on a transmission line, which is that no one still remembers high school trigonometry. As a result, trigonometric functions are kind of a pain to work with. We sidestep that mathematical headache using something called an analytic representation, where we let complex exponentials stand in for sinusoids. You can see that we go from our real-world representation to an analytic representation by replacing cosine of x with e to the j x. We get back to the real world by taking the real part of our analytic representation. However, we often don’t really need to go back to real representations of voltages because the analytic representation is sufficient for us to keep track of the phase of sinusoids on the line, and that’s mostly what we care about when we’re doing transmission line math.

Summary

- Sinusoidal drives result in sinusoids on transmission lines
- Frequency at any point on the t line is ω
- The wave number k describes “spatial frequency” at one time.

$$k = \frac{\omega}{v} = \frac{2\pi}{\lambda}$$

- We’ll substitute e^jw for cos(jw) often, an analytic representation.

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The Propagation Constant for Lossy Transmission Lines

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7

In this video we're going to try to figure out how loss in transmission lines affects sinusoidal signals

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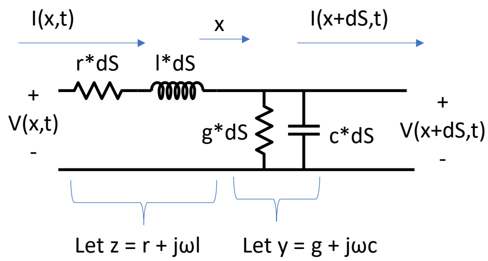
Find V(x,t) from V(0,t) with RLGC Divider



$$V(x+dS,t) = V(x,t)\frac{\left(\frac{1}{ydS}||Z_0\right)}{\frac{1}{ydS}||Z_0 + zdS}$$

$$V(x+dS,t) = V(x,t)\frac{\left(\frac{Z_0}{ydS}\right)/(Z_0+\frac{1}{ydS})}{\left(\frac{Z_0}{ydS}\right)/(Z_0+\frac{1}{ydS}) + zdS}$$

$$V(x+dS,t) = V(x,t)\frac{\left(\frac{Z_0}{ydS}\right)}{\left(\frac{Z_0}{ydS}\right) + zdS(Z_0+\frac{1}{ydS})}$$

$$V(x+dS,t) = V(x,t)\frac{Z_0}{Z_0 + yz(dS)^2 Z_0 + zdS}$$

$$V(x+dS,t) = V(x,t)\frac{1}{1 + zdS/Z_0}$$

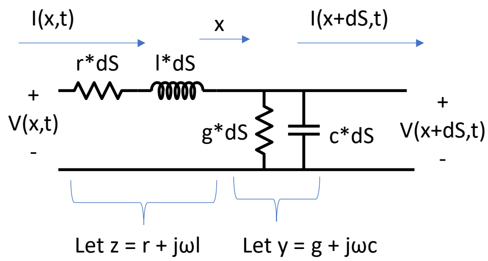
8

I promised we'd revisit modeling loss and dispersion once we put sine waves on transmission lines, and that time has come. We can take a closer mathematical look now because the analytic representation of our wave makes the math easier. Our strategy is to figure out how the voltage divider in the RLGC differential wire model changes the voltage of a differential, and then use that to form a differential equation the describes changes in the line voltage with position.

CLICK To make the math easier on that, we're defining two variables to simplify algebra. We're letting little z equal r plus jwl and little y equal g plus jwc

CLICK We set up our voltage divider equation here. Like before, we represent our shunt elements as one over admittance, and we put the shunt elements in parallel with Z0 to represent the impedance of the rest of the line. CLICK After that we do a couple of steps of exhausting algebra. Pause the video and do them by yourself if you'd like to keep up. CLICK the one notable step in the algebra is canceling out the dS^2 term because a differential squared is _really_ small. This step is sometimes called ignoring a second order term, or taking the equation to first order. CLICK Finally, we round out this page with a bit of factoring.

Find V(x,t) from V(0,t) with RLGC Divider



$$V(x+dS,t) = V(x,t)\frac{1}{1+zdS/Z_0}$$

$$V(x+dS,t) \approx V(x,t)\left(1-\frac{zdS}{Z_0}\right)$$

RECALL $\quad Z_0 = \sqrt{\frac{r+j\omega l}{g+jwc}} = \sqrt{\frac{z}{y}}$

$$V(x+dS,t) = V(x,t)(1-\sqrt{zy}dS)$$

$$\frac{V(x+dS,t)-V(x,t)}{dS} = V(x,t)\sqrt{zy}$$

$$\frac{\partial V(x,t)}{\partial x} = V(x,t)\sqrt{zy}$$

$$V(x,t) = V(0,t)e^{-\sqrt{zy}}$$

9

Picking up where we left off, CLICK we can take advantage of the fact that dS is small by using a binomial expansion of our equation. CLICK Then we remember our expression for Z0 and observe that it can be expressed in terms of z and y. CLICK And we substitute that into our expression to get something something tractable. CLICK We can move some terms around to reveal a deriviative in the limit as dS goes to zero.

CLICK Whew. That was lot of work for a rather simple equation at the end of the day. This is a first order differential equation, so you solve it by separating variables just like a time constant or an exponential growth equation. CLICK The result is that some initial voltage value V(0,t) gets scaled by an exponential as you move along x. I chose to represent this in terms of an initial voltage at x=0 becaue it makes for a clean equation, but you could write this equation for any x=x0 on the line and scale exponentially from that spot. The constant in this equation is just some initial condition.

Before we leave this slide, I want to re-emphasize the main takeaway. We've found a really useful equation; this last equation lets us find voltage anywhere on a transmission line If we know the voltage at one spot. Specifically, we find voltages at other parts of the transmission line, by scaling our known voltage exponentially.

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Complex Values Cause Attenuation,Dispersion

$\gamma = \sqrt{zy} = \alpha + jk$ — Define propagation constant γ, it is generally complex

$V(x,t) = V(0,t)e^{-\gamma x} = V(0,t)e^{-\alpha x}e^{-jkx}$ — α causes attenuation, k adds phase w/ x: it is wavenumber
(β is a common synonym for k in propagation constants)

Tedious math gets us here

$$\alpha = \sqrt{\frac{1}{2}\left(\sqrt{\omega^4(lc)^2 + \omega^2\left((lg)^2 + (rc)^2\right)} + (rg - \omega^2 lc)\right)}$$

$$k = \sqrt{\frac{1}{2}\left(\sqrt{\omega^4(lc)^2 + \omega^2\left((lg)^2 + (rc)^2\right)} - (rg - \omega^2 lc)\right)}$$

Group delay shows that dk/dω causes dispersion

$$\tau_g = -\frac{d\phi(\omega)}{d\omega}$$

Group delay comes from frequency dependent phase

10

We need to look closer at the exponential scaling factor we've identified to see what causes attenuation and dispersion. First, we're going to rename it because sqrt(zy) is a mouthful. We call this exponential scaling factor the propagation constant and give it the symbol gamma. We also note that it can be a complex number, so we often express it as alpha plus jk.

CLICK if we substitute alpha plus jk into the expression from the previous page, we find something interesting. alpha, the real part of gamma, causes attenuation. It makes signals get exponentially smaller with distance. k, the complex part of gamma, just looks like it adds phase with distance. That's the exact same thing that the wavenumber k did when we looked at infinite transmission lines, and that's because this k is the wavenumber! We still see phase advancing as we move along lossy transmission lines, and the k part of the propagation constant reflects that.

I need to add a few naming convention notes about the propagation coefficient. First, sometimes the symbol beta is used for the complex part of the propagation coefficient instead of k. Second, sometimes alpha is expressed as the tangent of an angle called the loss angle.

CLICK back on track, we can substitute expressions for z and y into our expression for gamma, and then do some truly heinous complex algebra to get expressions for alpha and k in terms of r, l, g and c. These expressions are horrendous, but they reveal something important, specifically, that k varies with omega in the general case. CLICK We don't have time to get into it here, but the delay of a sinusoid of some frequency through a system, which is called group delay, is given by the derivative of phase with respect to frequency. Having a value of k that depends on frequency gives rise to different delays at different frequencies, which is how you see dispersion in a line.

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α and β are Easier in Special Conditions

- Lossless condition (r=g=0) $\alpha = 0$ $k = \omega\sqrt{lc}$
- Low loss condition (jωl<<r, jωc<<g) $\alpha = \frac{r}{2Z_0} + \frac{gZ_0}{2}$ $k = \omega\sqrt{lc}$
- Heaviside condition (rc=gl) $\alpha = \sqrt{rg}$ $k = \omega\sqrt{lc}$

- k=ω/v because v=1/sqrt(lc), consistent with other videos.
- α has units of dB/m on most datasheets.
- These conditions have no dispersion, -d(-kx)/dw=sqrt(lc), a constant

Finally, we can simplify our expressions for the propagation constant quite a bit in a few special cases. The lossless condition is predictably easy. Note that k is equal to omega over v in this condition (and all of the conditions on this page) because v is equal to one over the square root of lc. We can find simple expressions for alpha in the low loss and Heaviside conditions too, which is nice. Note that alpha has units of inverse meters, which you can see because it's given by ohms per meter divided by ohms. That's because alpha gets multiplied by a distance to form the argument of an exponential, and exponential arguments need to be dimensionless. Inverse meters is a funny unit, and you won't see it on datasheets. Instead, alpha is commonly expressed as dB/m to make calculations easier.

Summary

- Propagation coefficient is used to find voltages at different places on the transmission line from voltage at a known position.

$$V(x,t) = V(0,t)e^{-\gamma} \quad = V(0,t)e^{-\alpha x}e^{-j\beta x}$$

- Attenuation comes from the real part of the propagation constant

- Dispersion comes from phase depending on frequency → group delay

- Special conditions have no dispersion. We'll mostly use these.

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Reflections of Sine Waves

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13

In this video we're going to find out what the voltage on a transmission line looks like when a sine wave gets reflected.

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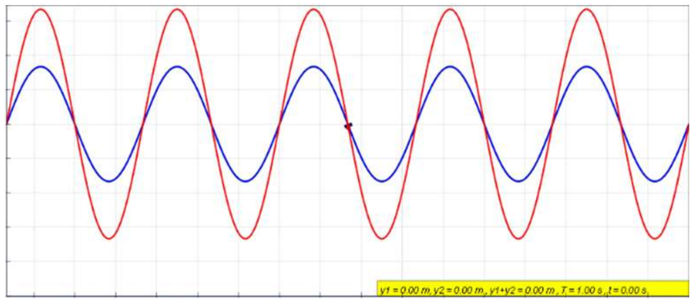
Sinusoids Reflecting Makes Standing Waves



- Standing waves are waves with static nodes and antinodes.
- Intuition: right-travelling and left-travelling velocity cancel.

https://commons.wikimedia.org/wiki/File:Waventerference.gif

And I'm going to cut straight to the answer here. The sum of a right travelling wave and a left travelling reflected wave is a standing wave pattern. This rather hypnotic graphic shows that: one wave is moving right, one wave is moving left, and the sum of them is standing still.

To give this a bit more formality, standing waves are waves that always have their peaks and troughs at the same position in x. Those positions are called nodes and antinodes. That's kind of a weird behavior, but I can rationalize it in my head if I remember that the right-going and left-going velocities of the component waves are equal and opposite. I think of the velocity of the component waves adding up, which makes the "motion" of the standing wave pattern cancel out.

This isn't a made-up example. This exact behavior could be seen in a transmission line terminated in an open circuit. Gamma in that case is one, so the reflected wave amplitude is the same as the incident amplitude, which would result in these equal-sized sinusoids running back and forth over one-another.

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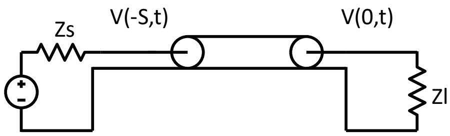
Envelope of Standing Waves Changes w/ 2k

$V_{in}(t) = V_{zp} e^{j\omega t} e^{-jk}$



$V(x,t) = V_+(x,t) + V_-(x,t)$ General solution

$V(x,t) = \frac{Z_0}{Z_s + Z_0}\left(V_{zp} e^{j(\omega t - k\ \ )} + \Gamma V_{zp} e^{j(\omega t + kx)}\right)$ Sub in analytic representation, reflection coeff

$V(x,t) = V_{zp} e^{j\omega t} \frac{Z_0}{Z_s + Z_0}\left(e^{-jkx} + \Gamma e^{jkx}\right)$ Separate t and x dependent terms, see Vin(t)

$V(x,t) = V_{zp} \frac{Z_0}{Z_s + Z_0} e^{j\omega t} e^{-jkx}\left(1 + \Gamma e^{2jkx}\right)$

$|V(x,t)| = V_{zp} \left|\frac{Z_0}{Z_s + Z_0}\right| \left|1 + \Gamma e^{-2\ \ \ }\right|$ Envelope evolves as 2kx, twice as fast as wave

Envelope max is amplitude times 1+|Γ|, min times 1-|Γ|

15

We'd like to get a good mathematical description of this standing wave pattern to see if it has any cool properties. To do that, we're going to find the voltage on a transmission line and then take it's magnitude to get the shape of the standing wave envelope. I've drawn a generic transmission line up here to start that process. I've made one sneaky change to the transmission line, which is that the load is now at position zero and the source is at position minus S. This change makes it easier to measure changes relative to the point of reflection. Specifically, it means that setting x=0 in the line voltage will tell us the behavior at the load. We have also arbitrarily chosen to define the load as having zero phase, which means we need to add a constant phase offset to the input so that the phase picked up over the line is reduced to zero at the load. That's just a choice of mathematical definition, it doesn't reflect changing the design or physics of the input source at all.

CLICK We're going to start by noting the standing wave pattern arises from a general solution to the wave equation. So we write that V(x,t) is the sum of right-travelling and left-travelling waves.

CLICK Then we substitute in functions that represent the waves. We know the right-travelling wave is going to be e to the j omega t minus k x because of our work with infinite transmission lines: each point on the line has a phase shifted version of the input source living on it. We could have instead used the propagation coefficient to produce the same equations : each point on the line will be e to the minus gamma x multiplied by the input voltage. Finally, we know the left-travelling wave is going to be generated by a reflection, so it must be Gamma times the right-travelling wave.

CLICK We can do some factoring to reveal interesting behavior. First, we factor out the omega term to separate the time and space behaviors of the line. This results in the Vin(t) time-varying sinusoid appearing at the front of the line offset by the phase of the transmission line. Vin(t) is being multiplied by the divider between Zs and Zdp.

CLICK Finally, we take the magnitude of V(x,t) to find the size of the envelope around our standing wave at every x. e to the j x has a magnitude of one regardless of the value of x, so e to the j omega t and e to the minus j k x fall out of our expression. Said another way, we don't care about these simple phase shifts when we're looking at magnitude, and that's good because our standing wave doesn't seem to have phase shifts in it that depend on time.

The interesting term in the magnitude expression is this last one, which says that the wave envelope is going to be scaled by a factor that depends periodically on x. This factor has twice the wave number as each individual wave, so we expect two nodes in each wavelength of a single wave. This makes sense if you reflect on the velocity of the right-travelling and left-travelling waves again: they'll hit one another twice as fast because of their opposite velocities.

If Gamma is real, the maximum the factor can be is when e to the minus 2jkx is one, so the largest value 1+Gamma. The minimum is when e to the minus 2jkx is minus one, so the smallest value is 1 minus Gamma. Complex Gammas will have some phase shift relative to the purely real 1 they're being added to, but they can't add or subtract more than their magnitude for any phase of e to the minus 2jkx exponential. You can think about that by imagining 1 and Gamma e to the minus 2jkx as vectors in a complex plane.: Gamma e to the minus 2jkx launches of (1,0) on the complex plane, and no matter how you rotate that vector, it can never add or subtract more than its magnitude from the real number 1.

As a final comment, we included the source divider in this expression, and the source divider _could_ be complex if either Z0 or Zs has an imaginary part, but it's still just telling us how much voltage gets onto the line at x equals minus S.

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Summary

- The sum of forward and reverse sine waves is a standing wave.
- The envelope of the standing wave has twice the wave number of the waves being added.
- The max value of the envelope is amplitude*(1+|Γ|) and the minimum is amplitude*(1-|Γ|)

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Voltage Standing Wave Ratio

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17

In this video we’re going to see a few examples of standing waves, and we’re going learn about a metric that helps describe standing waves.

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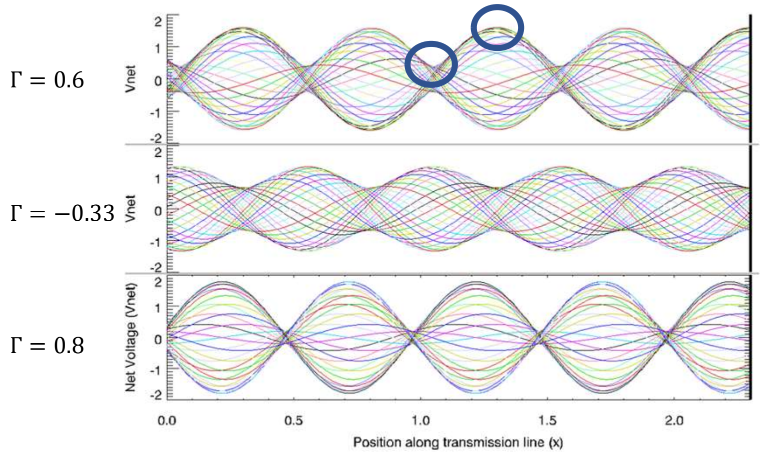
VSWR is Ratio of Standing Max V to Min V



$$|V(x,t)| = V_{zp}\left|\frac{Z_0}{Z_s + Z_0}\right|\left|1 + \Gamma e^{-2jkx}\right|$$

$$VSWR = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

- VSWR: voltage standing wave ratio

18

I've included a figure from Wikipedia here that shows some standing waves for different values of Gamma. I've also the equation for the envelope of a standing wave. The first thing I want you to notice is that the maxima and minima of these standing wave patterns do seem to depend on Gamma in the way we predicted. We thought the maxima would have a height of 1+Gamma times the amplitude, which is 1V in this case, and the minima would have a height of 1-Gamma times the amplitude. We can se that CLICK this peak is at 1.6V and CLICK this trough is at 0.4V, which lines up with expectations.

As an aside, the middle graph has an interesting feature, which is that the reflection coefficient contributes some phase to the reflected wave. A Gamma of -0.33 is still purely real, but the negative sign means reflected waves have an additional 180 degrees of phase compared to incident waves. This is a reminder that reflection coefficients, especially complex reflection coefficients, add phase to reflected waves.

VSWR, or voltage standing wave ratio, is the ratio of peak height to trough height. So this point divided by this point. Because we know the peak and trough voltages have the same scaling factor in front of them, and we already know the maximum and minimum value of the x-dependent factor in our envelope, it's pretty easy to calculate their ratio. CLICK We find that VSWR is the ratio of 1 plus the magnitude of Gamma over 1 minus the magnitude of Gamma.

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VSWR is Interchangeable with Γ

$$VSWR = \frac{1+|\Gamma|}{1-|\Gamma|} \longleftrightarrow |\Gamma| = \frac{VSWR-1}{VSWR+1}$$

- Datasheets will list one or the other depending on vendor
- VSWR > 5 is very hard to work with.

19

You might think the VSWR equation we have is a bit silly because VSWR tells us exactly the same thing as the magnitude of Gamma. A little bit of algebra will reveal that you can write a function to go either VSWR to magnitude of Gamma or from magnitude of Gamma to VSWR. So why care about both numbers, particularly if we could know the full complex value of Gamma, which tells us more than VSWR?

The answer is historical. VSWR is important because it is relatively easy to measure using a device called a slotted line, which you can imagine as a transmission line with a sliding oscilloscope probe on top that can measure the magnitude of V(x,t). Old-timey engineers relied on VSWR before we were able to accurately measure complex valued quantities like Gamma. That heritage continues today because VSWR still shows up on many datasheets.

A few final notes: First, VSWR can provide a handy rule of thumb for when an RF system is too reflective. Engineers I've spoken to consider VSWR>5 essentially unusable. Second, if you're talking to an old-timey engineer they might insist VSWR is pronounced vizwer. That's a matter of taste.

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Summary

- VSWR is the ratio of the voltage at the peak of the standing wave to the voltage at the trough
- VSWR says the same thing as $|\Gamma|$

$$VSWR = \frac{1+|\Gamma|}{1-|\Gamma|} \longleftrightarrow |\Gamma| = \frac{VSWR-1}{VSWR+1}$$