

Lecture 02:  
Intro to Transmission Lines

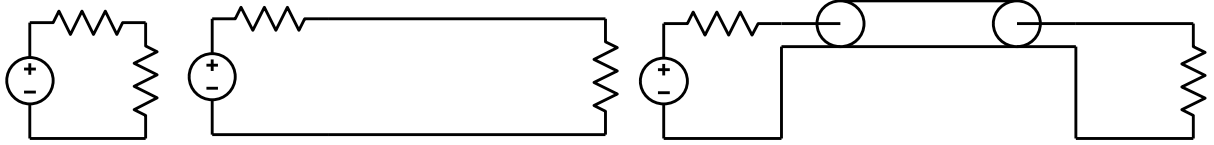
# Introducing the Transmission Line

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E157 – Radio Frequency Circuit Design

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In this video we're going to introduce a new circuit element called transmission line

## RF is Special Because We Care About Delay



These are the same  
(but maybe not at RF?)

Transmission Line!

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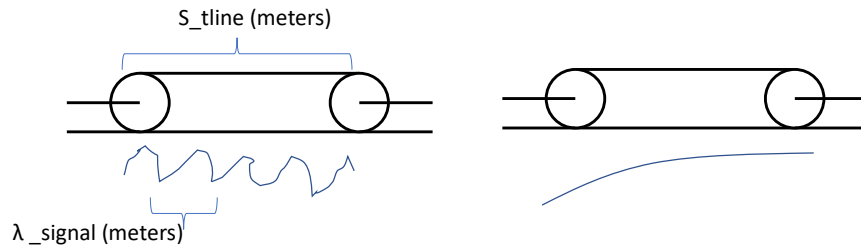
We need this new circuit element because you care about the delay signals experience when signals are changing quickly.

To give that statement some context, consider that our existing circuit analysis says that these two circuits are exactly the same, but the long wire in the second circuit can make a difference at high frequencies or short time scales.

So we indicate that we care about the speed of a wire using a special symbol called a transmission line.

(Aside: The relative Z of parasitics is also funny at RF, more later)

## Use T Line Models If $\lambda_{\text{signal}} \ll S_{\text{tline}}$



- If  $\lambda_{\text{signal}} \gg S_{\text{tline}}$ , voltage approx. same on both sides of tline
- Why no T Lines in E80? ( $f_{\text{signal}} \sim 100\text{Hz}$ ,  $v_{\text{signal}} \sim 0.3c$ )
- Rule of thumb: need t lines if  $\lambda_{\text{signal}} > S_{\text{tline}}/10$ .

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You've been using wires your whole career, but you've probably never used a transmission line model. How did we fail to notice all these transmission lines everywhere?

The figure here has a hint: if transmission lines are short compared to the wavelength of our signals, then the voltage on both sides of the transmission line is the same. On the other hand, if the transmission lines are long compared to the signal's wavelength, then the voltage on one side of the transmission line could be very different from the other.

As a more concrete example, why didn't we use transmission line models in E80? Signals in that class were  $\sim 100\text{Hz}$  at longest, and let's say the velocity of signals in E80 wires was  $1/3$  the speed of light. Pause the video and try to figure out why we didn't care about transmission lines.

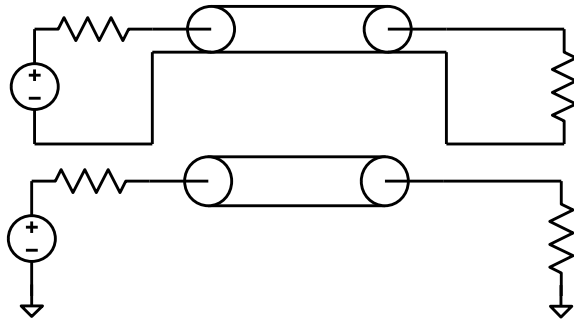
$\lambda_{\text{E80}} \sim 10^6 \text{ m}$  by  $v = \lambda f$ , much longer than our wires! So we didn't need to care!

The usual rule of thumb is that you need to include transmission lines if your signal wavelength is more than  $\sim 1/10$ th the length of your t line.

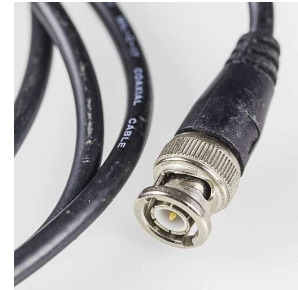
Aside: Square waves have deceptively long wavelengths ... (Note:  $\lambda_{\text{square}} \sim 5 \times \lambda_{\text{sine}}$ ) ... mention rise time? Cut it for now.

## T Line Models have Signal & Return Terminals

Physically insightful



Looks Better



- Both signal and ground are hooked up to T Line, like a BNC cable.
- Ground shield return paths matter because they cause inductance.
- Each pair of terminals is called a port, these have two ports.

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Transmission lines are a little bit weird compared to circuit elements you have seen because they have four terminals.

Two represent the signal pins of a transmission line on either side (like this spot on the BNC) and two represent the ground connections on either side (like this spot on the BNC). The ground connections are important because they are the return path for current in the circuit, and the exact path current takes creates inductance, which is impactful at high frequencies. Also, note that we're using a BNC cable as our example transmission line here, but there are lots of types of transmission line in the world: SMA cables, UFL cables, certain types of PCB traces, etc. Further note that not all wires are transmission lines, we'll get more specific about what makes a transmission line special later.

Most circuit elements you've dealt with have two terminals, so they are called one-port devices. A pair of terminals you measure across is called a port. Transmission lines are two port devices, which will matter later.

Sometimes we are going to get lazy and skip drawing the connection to the ground shield because it makes the diagram simpler, but you can assume it's always there.

## Summary

- We represent delay in RF circuits using transmission lines
- Use transmission line models if  $\lambda_{\text{signal}} > S_{\text{tline}}/10$
- We sometimes leave return paths off our symbols, but they matter!

# Models of Wires

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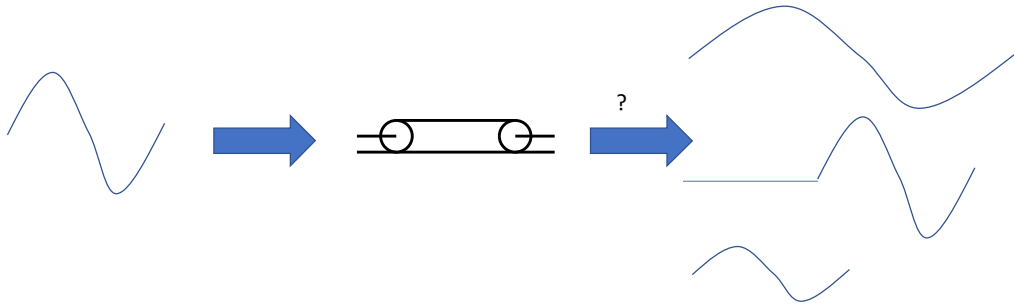
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In this video we're going to try to understand transmission lines by making circuit models for wires.

## Can We Make a Circuit Model of Delay?



- How are signals affected by a T. Line: Stretch? Shift? Scale?
- Try to understand T. Line by making model of what's happening in it.

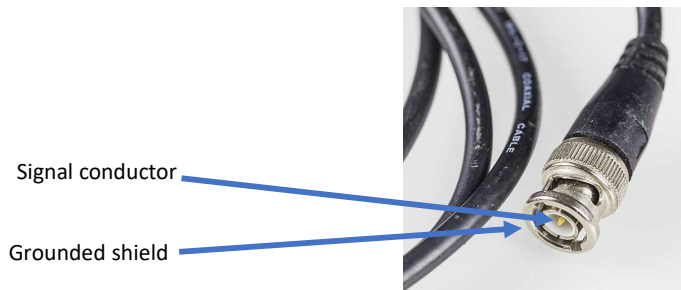
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We need to do this because we don't have a great model of what a transmission line does right now. Since we don't have a mathematical model of a transmission line, it could turn the signal on the left into any of the signals on the right.

We don't have many clues yet, so we're going to start by making circuit models of wires.



## Basic Facts: Wires have Series R, Shunt C



- The conductor is made of metal, so it has resistance.
- The shield is also metal & charge between two metals is capacitance.
- Shunt is vocabulary: it means hooked “in parallel” to ground.

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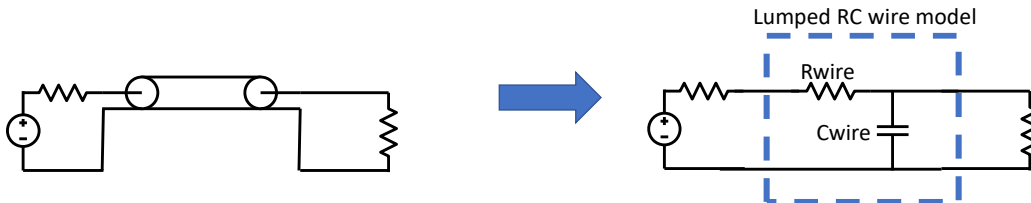
It's easiest to make circuit models of wires if you start with a few basic facts.

First, wires are made of metal, so they have some resistance. We expect to see a resistance in series between the signal connections at either end of a transmission line.

Second, transmission lines always have both a conductor and a return path. That means they are made of two pieces of metal that are usually close together. Charge building up on pieces of metal that are close together makes a capacitance, so there is capacitance between the conductor and the return path.

Because our return path is grounded, the capacitance will have to be hooked from our signal to ground, which we call being “in shunt”

## Lumped RC is a Simple, Pessimistic Model



- It's a delay: sine waves have phase shift, square waves have rise time
- It's pessimistic: we imagine all the wire cap "lumped" at one end

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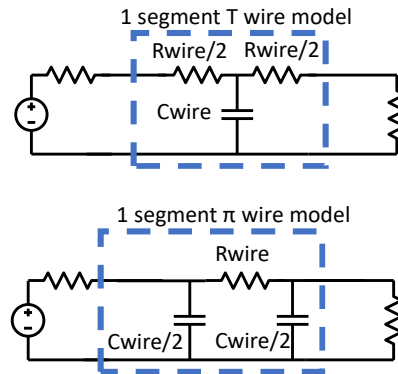
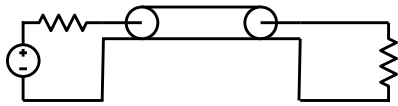
The simplest circuit model we could imagine that matches our basic fact is a lumped RC model: just replace the wire with a series resistor and a shunt cap on one side or the other.

This model does make a delay. Sine waves driven into this wire would have a phase shift, which means their zero crossings would come later. And square waves would have a rise time, which shifts signals to the right.

However, this model has a few problems. We're going to focus on one, which is that it's pessimistic. We imagine all of the capacitance in the wire comes after all of the resistance in the wire, when they're actually smeared together.

When we treat all of the capacitance or resistance of an element as happening at one place, we call it a lumped model. All of the circuits you've seen so far use lumped models.

## T or $\pi$ RC Models are Surprisingly Good



- Used for digital design
- You can do even better by adding more segments
- “Distributed” models use infinitely small segments  $\rightarrow$  differential eqn

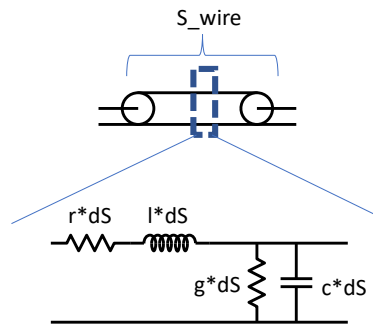
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We can try to make lumped models work for us a little better by making something called a T or pi model. In T models, the capacitance is put in the middle of the wire resistance, in pi models the capacitance is cut in half and put on either side of the wire resistance.

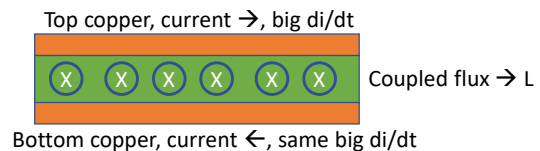
These models actually do a really good job of predicting delays for digital designs, and they're good for hand analysis.

However, if you want to be really accurate, you could make the wire out of more T and pi segments, each of which had less resistance and capacitance. If you eventually make the segments infinitely small, you wind up with a “distributed” model, which describes voltages and currents with differential equations.

## Differential RLGC Models Used for RF Design



Source of L, imagine a PCB trace



- Note that differential model uses  $r = R/S_{\text{wire}}$  , etc.
- L comes from current return path coupling flux between wires
- G represents dielectric leakage, usually small

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The wire models we use for RF design are called distributed RLGC models, and they have a few more elements than the T or pi models.

Each differential element of the wire is composed of a tiny series resistor, series inductor, shunt capacitor and shunt conductance. The lower case letters indicate the value per unit length of the wire, and  $dS$  is the length of a differential segment of the wire, so the differential resistance is given by the resistance per unit length times  $dS$ .

We need to add the inductance because return currents and forward currents both create magnetic fields, and fields coupled between conductors is how you get inductance. As an example, consider a wire on a PCB above a ground plane: current will go out through the wire and return through the ground plane, and applying the right hand rule shows that they couple magnetic field between the two wires.

We need to add the conductances because the capacitors used to make wires aren't perfect, and sometimes they leak some current.

## Summary

- “Lumped” models sum many small circuit elements into a big one
- “Distributed” models cut a circuit into differential pieces
- Differential RLGC models are used for RF calculations.
- Aside: you can get away with RC T or  $\pi$  models for digital design

# Telegrapher's Equation

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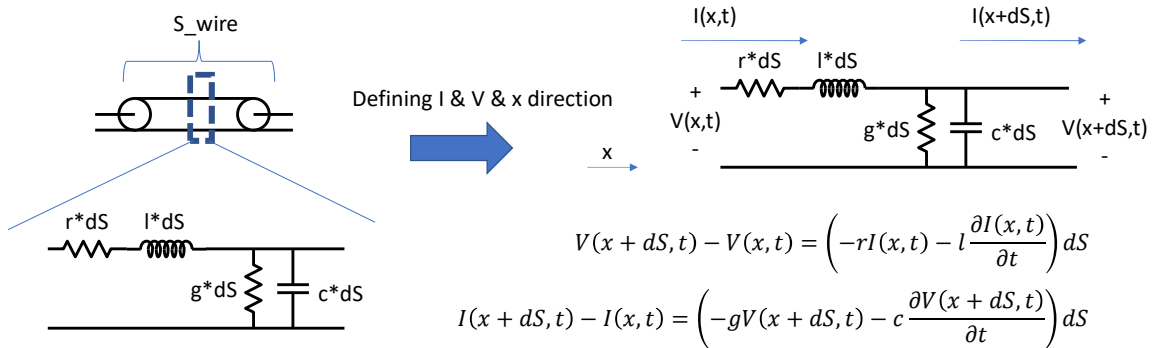
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In this video we're going to construct an equation for how voltage and current move in a transmission line.

## Distributed RLGC Models Make a Diff Eqn



- Use KVL to make first equation and KCL to make the second.

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To do this, we're going to start from the distributed RLGC wire model and punch it up by defining a spatial coordinate, and voltage and current variables on either side of our differential piece of wire. We're going to call the coordinate along the length of the wire  $x$ . The whole wire has a length  $S_{\text{wire}}$ , and each circuit element is a value per unit length multiplied by the length of our tiny segment,  $dS$ . We call the voltage on the left side of our differential  $V(x,t)$  and the right side  $V(x+dS,t)$ . Current is named similarly:  $I(x,t)$  on the left and  $I(x+dS,t)$  on the right.

We can write a Kirchoff's voltage law equation for this differential. The right side of the wire is at a lower voltage than the left side, and it shows the voltage drop between the right and left is due to the current in the resistor and inductor. So this equation shows negative resistive and inductive voltage contributions to the voltage difference between  $V(x+dS,t)$  and  $V(x,t)$ .

We can also write a Kirchoff's current law equation. Any current that doesn't make it from the left side of the wire to the right has to be lost through the conductor and the capacitor. So we see a negative conductive and capacitive contribution to the current difference between  $I(x+dS,t)$  and  $I(x,t)$ .

These are coupled partial differential equations. Coupled because they each have  $I$  and  $V$

variables in them, and partial because there are derivatives with respect to both time and space. We'll need to decouple them to solve them.



## Separating the Diff Eqn Leads to a Wave Eqn

$$V(x + dS, t) - V(x, t) = \left( -rI(x, t) - l \frac{\partial I(x, t)}{\partial t} \right) dS \quad I(x + dS, t) - I(x, t) = \left( -gV(x + dx, t) - c \frac{\partial V(x + dx, t)}{\partial t} \right) dS$$

$$\frac{\partial V(x, t)}{\partial x} = -rI(x, t) - l \frac{dI(x, t)}{dt} \quad \frac{dI(x, t)}{dx} = -gV(x + dS, t) - c \frac{\partial V(x + dS, t)}{\partial t}$$

$$\frac{\partial^2 V(x, t)}{\partial x^2} = -r \frac{\partial I(x, t)}{\partial x} - l \frac{\partial}{\partial x} \frac{\partial I(x, t)}{\partial t}$$

$$\frac{\partial^2 V(x, t)}{\partial x^2} = -r \left( -gV(x + dS, t) - c \frac{\partial V(x + dS, t)}{\partial t} \right) - l \frac{\partial}{\partial t} \left( -gV(x + dS, t) - c \frac{\partial V(x + dS, t)}{\partial t} \right)$$

$$\frac{d^2 V(x, t)}{dx^2} = rgV(x, t) + (rc + gl) \frac{dV(x, t)}{dt} + lc \frac{d^2 V(x, t)}{dt^2} \quad \leftarrow \text{The Telegrapher's Equation, a partial diff eqn}$$

$$\frac{d^2 V(x, t)}{dx^2} = lc \frac{d^2 V(x, t)}{dt^2} \quad \leftarrow \text{The Lossless Telegrapher's Equation, a wave equation! (we let } r=0 \text{ ohm/m \& } g=0 \text{ S/m to get here)}$$

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We're going to do that on this slide. I've copied the equations from the first slide. This is going to be a bit mathy, but I wanted to include this derivation because it's a great example of how to decouple partial differential equations. Steps like this for electromagnetic wave equations and pressure wave equations too, so it's good to have seen them. I invite you to work through these steps on paper with me, and don't be afraid to pause to make sure you're getting the same result.

Clicks:

First, we recognize that if we take  $dS$  to the limit, we have a derivative with respect to  $x$  in both equations

But we can't substitute one into the other right away because the current equation is in terms of  $dI/dx$  instead of  $I$ . So we take a derivative of the left equation to get it in terms of  $dI/dx$ .

Then we substitute the equations together and do a trick that mathematicians keep warning me about: we swap the order of the derivatives on the second term

Gathering terms together, we wind up with an equation called the Telegrapher's Equation. As the name suggests, this was super important for early telegrams. It describes how voltage pulses move in wires.

There's one more important simplification that is commonly made, which is assuming that our transmission line is lossless. Doing so leaves us with a special partial differential

equation called a wave equation.

Wave equations are super important, and we'll talk about solutions to them in another video.

## Summary

- You can construct coupled diff eqns from the RLGC model
- Combining them & ignoring loss gives us a partial diff eqn:

$$\frac{d^2V(x,t)}{dx^2} = lc \frac{d^2V(x,t)}{dt^2}$$

- This is a wave equation

# Wave Equations

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In this video we're going to talk about wave equations and how to solve them.

## Wave Equations are Common and Important

$$\frac{\partial^2 p}{\partial x^2} = \frac{\rho_0}{B} \frac{\partial^2 p}{\partial t^2} \quad \text{Pressure / velocity waves (sound, water)}$$

$$\frac{\partial^2 E}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} \quad \text{E field / B field waves (vacuum EM)}$$

$$\frac{\partial^2 V}{\partial x^2} = lc \frac{\partial^2 V}{\partial t^2} \quad \text{Voltage / current waves (t lines)}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \quad \text{In general}$$

- Most commonly arise w/ “cross coupled” partial diff eqn. eg: Maxwell
- $v = 1/\text{sqrt}(lc)$  for a t line.

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To motivate that, this slide has examples of different wave equations. The wave equation is used to describe sounds, light, quantum mechanical probability, and many other phenomena, so they're an important class of equations.

You'll notice that all of the example wave equations I've included have the same form: a second derivative with respect to space on the left, a second derivative with respect to time on the right, and a proportionality constant which is called  $1/v^2$ . We'll talk more about  $v$  in a minute, but note that  $v=1/\text{sqrt}(lc)$  for transmission lines

## Wave Equations Solved by $f(x-vt)$ or $f(x+vt)$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2}{\partial x^2} f(x - vt) = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} f(x - vt) \quad \text{Substitute candidate solution}$$

$$\frac{d^2 f(x - vt)}{d(x - vt)^2} = \frac{1}{v^2} v^2 \frac{d^2 f(x - vt)}{d(x - vt)^2} \quad \text{Chain rule}$$

- Solve differential equations with guess and check
- Both  $f(x-vt)$  and  $f(x+vt)$  are solutions, superposition applies
- There's an alternate form of solution we'll use later:  $f(kx \pm \omega t)$

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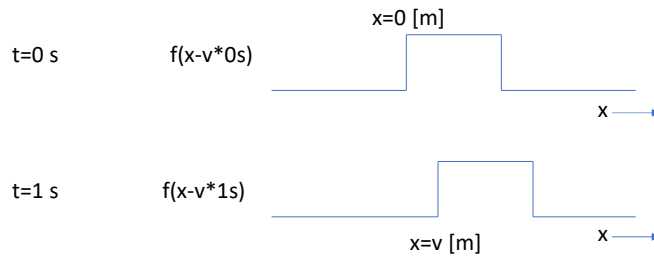
We're going to solve wave equations like any other differential equations, by using guess and check.

We're going to use a guess that wave equations are solved by  $f(x-vt)$ , where  $f$  is any function. We substitute this guess into our wave equation, and we see that evaluating the guess gives a self-consistent answer. Note that we used the chain rule to convert the partial derivatives to total derivatives between steps two and three.

That means  $f(x-vt)$  is a candidate solution!  $f(x+vt)$  would also work using the exact same logic. Because this is a linear equation, our general solution is going to be a scaled sum of all candidate solutions.

Also, note that this there's another form for this candidate solution,  $f(kx + \omega t)$ , where  $k = \omega/v$ . We'll use this alternate form later,  $k$  is called the wave number.

## Wave Solutions Propagate at Speed $v$



- A property of  $f(x-vt)$  is that a  $f(x,0)$  will propagate to the right.
- $f(x-vt)$  /  $f(x+vt)$  are referred to as right / left travelling waves.

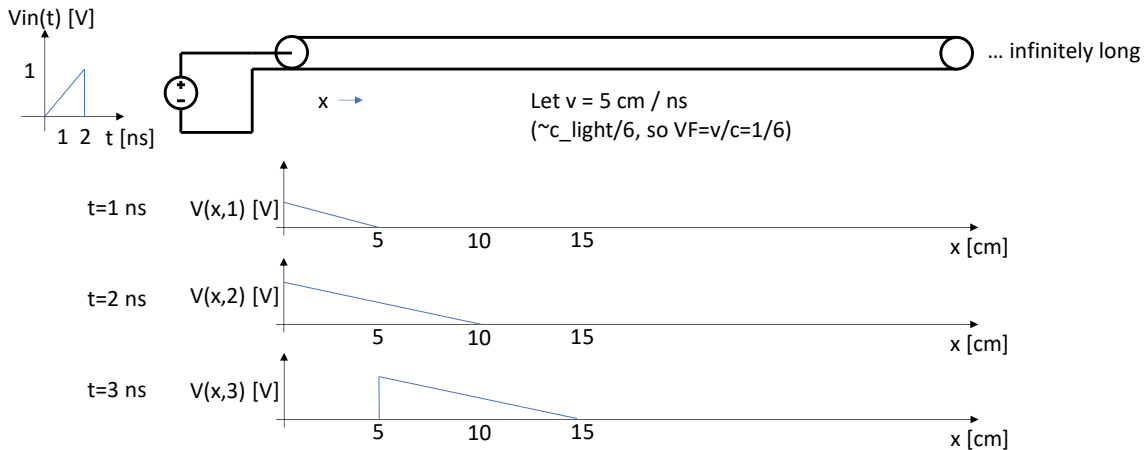
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This solution might seem dissatisfying, but it does one cool wave thing: all of these solutions propagate.

To see this, consider some function  $f$  at time zero, like in this top picture. 1s later, it would have changed from  $f(x)$  to  $f(x-v)$ , so it would have shifted to the right by  $v$  meters because we have subtracted  $vt$  from the argument of the function. The  $vt$  term causes solutions to propagate to the right at a speed of  $v$ .

This is backwards for the  $f(x+vt)$  solution, add  $vt$  causes the initial function to propagate left. Because of this, the two candidate solutions are often called the left and right travelling waves.

## The Function $f$ is set by Boundary Conditions



- No left-propagating wave in this example b/c no boundary creates it.

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Even though this solution is propagating like a wave, we still don't know how to pick the function  $f$ . It turns out that  $f$  is set by boundary conditions on the wave equation, and one good example is a voltage source driving a transmission line.

This schematic shows a voltage source driving a signal  $V(t)$  onto a transmission line. We're letting the speed of waves in the transmission line be 5 cm / ns, which is one sixth the speed of light. Because velocities are so fast, we often talk about them as fractions of the speed of light rather than big numbers. We do that using a variable called the velocity factor,  $VF$ , which is the ratio of the velocity in the line to the speed of light.

I have to clarify one bit of notation: The voltage source is driving a voltage  $V_{in}(t)$ , but up until now we've talked about a solution  $f(t)$ . On this slide we're going to replace the letter  $f$  in  $f(x-vt)$  with  $V$ , to make  $V(x-vt)$ .  $V(x,t)$  and  $f(x,t)$  are both solutions to wave equations, but we're using the symbol  $V$  here because we're talking about a voltage on a transmission line, instead of some arbitrary function that solves the general case.

At time zero and earlier, the voltage source is enforcing the voltage on the left side of the line to be 0V. At time=1ns, the voltage source has started enforcing a different voltage at the end of the line, and those voltages start to propagate to the right in keeping with the wave equation solution. The continues until the falling edge at  $t=2$ ns. After that, the



voltage enforced on the line will just keep propagating down it forever, like in the  $t=3\text{ns}$  case. So the shape of  $V(x,t)$  was determined by the boundary condition  $V_{in}(t)$  enforced on  $x=0$ ; this shape here, is  $f(x,t)$ .

This solution only has a right-propagating wave as its solution. However, other boundary conditions can create left propagating waves. We'll see those soon. Said another way: even though our general solution is the sum of a left-propagating and right-propagating wave, we won't see both candidate solutions for every boundary condition.

## Summary

- This is the general wave equation:  $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$
- Wave equations are solved by  $f(x-vt)$  or  $f(x+vt)$ : a superposition of left and right travelling wave solutions.
- Waves propagate at speed  $v$
- The function  $f$  in the solution is set by boundary conditions