Department of Engineering

Open Circuit Time Constants & Short Circuit Time Constants

Matthew Spencer
Harvey Mudd College
E151 – Analog Circuit Design

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In this video series we're going to continue our examination of amplifier dynamics and, in particular, bandwidth estimation. We're trying to improve on where we left off, which was calculating full transfer functions or using Miller Approximations. We're going to figure out some new analysis by getting back to basics with a model for the mid-band amplifier response, and then we're going to learn some fabulous circuit theory that lets us pick out the important features of that model. This fabulous circuit theory is called open circuit time constant analysis, and it has a lot of nice properties for a designer, notably that it gives us intuition about what capacitors matter in a circuit. We'll wrap up dynamics in the next video series by applying open circuit time constants to a few circuits.

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Modeling The Mid-Band Approximation

Matthew Spencer

Harvey Mudd College

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In this video we're going to write a generic transfer function that lets us model the dynamics almost any amplifier. We'll analyze the details of this model in future videos.

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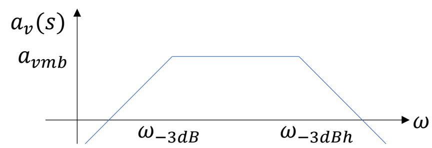
Transfer Function Dynamics =1 in Mid-Band



$$a_v(s) = a_{vmb} \cdot \frac{s^n \cdot \left(\frac{1}{p_{z1}p_{z2} \dots p_{zn}}\right)}{\left(\frac{s}{p_{z1}}+1\right)\left(\frac{s}{p_{z2}}+1\right) \dots \left(\frac{s}{p_{zn}}+1\right)} \cdot \frac{1}{\left(\frac{s}{p_1}+1\right)\left(\frac{s}{p_2}+1\right) \dots \left(\frac{s}{p_n}+1\right)}$$

$$a_v(s) = a_{vmb} \cdot \frac{s^m}{(s+p_{z1})(s+p_{z2}) \dots (s+p_{zm})} \cdot \frac{1}{(\tau_1 s+1)(\tau_2 s+1) \dots (\tau_n s+1)}$$

$$a_v(s) = a_{vmb} \cdot \frac{s^m}{s^m + s^{m-1}(p_{z1}+p_{z2}+\dots+p_{zm}) + \dots + p_{z1}p_{z2} \dots p_{zm}} \cdot \frac{1}{1+(\tau_1+\tau_2+\dots+\tau_n)s + \ \dots + \tau_1\tau_2 \dots \tau_n s^n}$$

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I've included a rough sketch of our mid-band response in this figure. It's a simplification, because I'm only showing one zero at the origin, one dominant pole at low frequency and one dominant pole at high frequency. In reality, there are more poles at frequencies above w-3dBh and more zeros and poles below w-3dBl. That's because there are a lot of caps in amplifiers, at least two per transistor, and each of those has the potential to create a pole and a zero. We'll include those in our equations, but I haven't opted to drawn them for reasons that will become clear in the next video.

CLICK We can represent this mess of poles and zeros by multiplying them together. The big trick with this transfer function is that we're trying to separate out gain and dynamics, which means we have to guarantee that the dynamics terms are 1 in the mid-band. The term on the very left is our mid-band gain, which is also sometimes called our DC gain because lots of real amplifiers aren't AC coupled. The term in the middle represents the rising slope on the left of our figure as our coupling network impedance drops and our bypass capacitors kick in. We imagine we have some number of zeros, m, at the origin, which is close enough because our coupling network doesn't let anything through at DC, and then we imagine that each of those zeros has an associated pole at pz1 through pzm. We need to have this product of 1/pzi on the top so that the whole middle term is equal to 1 in the mid-band. When s/pzi is bigger than one, each pole has a value of omega/pzi, and we need to cancel that in the numerator. The term on the right represents our roll off at high frequencies, and we model it as a bunch of poles at frequencies p1, p2 all the way up to pn. I've used n as the subscript for the poles here to show that in general you can have different numbers of high frequency poles on the right and low frequency poles on the left.

CLICK We're going to rewrite each of these terms in a slightly different format. I've written the high frequency poles in terms of their time constants instead of their poles, which is easy enough because each taui=1/pi. I've also opted to rewrite the low-frequency pole-zero pairs in canonical form instead of factor form, so each individual pole-zero pair is s/(s+p). This is a weird choice for me, because I think factor form makes frequency analysis much easier in general, but this canonical form will be useful in the future.

CLICK Finally, we're going to distribute the denominators, which will help with some approximations we're going to do later. And that's it! These are all the representations of our mid-band response that we're going to use. There's a coupling term with a bunch of pole-zero pairs, a bunch of of high frequency poles, and a mid-band gain.

… Modeling amplifier dynamics is hard because amplifier dynamics are complicated. There are a lot of poles and zeros in amplifiers. Even our simplest amplifier, the common emitter, has a right-half plane zero and two poles. And it has all of that stuff before we include the coupling network or the degeneration capacitance in a voltage divider bias. So we have to make some judicious simplifications to manage modeling it in a rational way. The main one we make is assuming that two poles dominate our mid-band response, one at w-3dBl and one at w-3dBh. There are more poles and zeros beyond these, but I haven't drawn them because we'll approximate them away soon.
… we've skipped analyzing coupling , though just adds a high pass filter in series
… our main problem is bandwidth estimation, our second issue is ensuring stability when we put amplifiers in feedback

Summary

- We can estimate bandwidth by finding the dominant high-frequency and low-frequency poles in our overall transfer function.

- The general transfer function for an amplifier with a mid-band has a DC term, low f dynamics and high f dynamics.

$$a_v(s) = a_{vmb} \cdot \frac{s^m \cdot \left(\frac{1}{p_{z1}p_{z2} \dots p_{zm}}\right)}{\left(\frac{s}{p_{z1}}+1\right)\left(\frac{s}{p_{z2}}+1\right)\dots\left(\frac{s}{p_{zm}}+1\right)\dots} \cdot \frac{1}{\left(\frac{s}{p_1}+1\right)\left(\frac{s}{p_2}+1\right)\dots\left(\frac{s}{p_n}+1\right)}$$

$$a_v(s) = a_{vmb} \cdot \frac{s^m}{s^m + s^{m-1}(p_{z1}+p_{z2}+\dots+p_{zm}) + \dots + p_{z1}p_{z2}\dots p_{zm}} \cdot \frac{1}{1+(\tau_1+\tau_2+\dots+\tau_n)s + \dots + \tau_1\tau_2\dots\tau_n s^n}$$

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Open Circuit Time Constants

Matthew Spencer

Harvey Mudd College

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In this video we’re going to learn about a way to pick useful features out of our model for the mid-band using a technique called open-circuit time constants.

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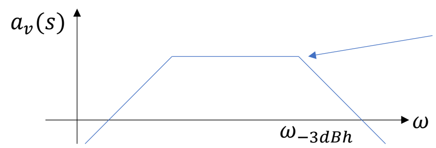
Dominant Pole Location Given by Sum of τ_i



Only need this for BW estimate, assume a dominant pole

At $s = j\omega_{-3dB}$, these terms are 1

$$a_v(s) = a_{vmb} \cdot \frac{s^m}{(s + p_{z1})(s + p_{z2}) \dots (s + p_{zm})} \cdot \frac{1}{(\tau_1 s + 1)(\tau_2 s + 1) \dots (\tau_n s + 1)}$$

≈0 because $\tau_i \omega_{-3dBh} < 1$ for $i >$ 2 (presuming a dominant pole)

$$a_v(j\omega_{-3dB}) = a_{vmb} \cdot \frac{1}{1 + (\tau_1 + \tau_2 + \dots + \tau_n) j\omega_{-3dB} + \dots + \tau_1 \tau_2 \dots \tau_n (j\omega_{-3dB})^n}$$

$$a_v(j\omega_{-3d}\ ) \approx a_{vmb} \cdot \frac{1}{1 + (\tau_1 + \tau_2 + \dots + \tau_n) j\omega_{-3dB}}$$

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Open circuit time constants, or OCTC, are a circuit feature that is part of a bandwidth estimation technique invented at MIT in the 1960s. Bandwidth estimates made with OCTC have lots of nice properties, like being conservative and granting intuition about which capacitors are slowing a circuit down. The heart of the technique is simplifying the mid-band transfer function model we just made.

CLICK Specifically, we simplify it by realizing that we only care about the high frequency pole, which is usually a dominant pole. Because that pole is dominant, modeling this as a first order system is reasonably safe, and we'll see that there's a way to do it easily and accurately.

CLICK The first simplification we're going to make is getting rid of the low-frequency pole-zero pairs because we're operating at a frequency well above any of the pzi. That means each pole-zero pair in this is equal to 1 because jw is bigger than the linked pzi.

CLICK That leaves us with this reduced transfer function in terms of the taui. Great! I've also expanded the denominator into a polynomial rather than leaving individual factors.

CLICK The expanded denominator lets us realize that all of these high order s terms are going to be pretty close to zero. That's because tau1 times w-3dBh is 1 by defnintion at the dominant pole, and every other taui is smaller than taui. Multiplying those terms by w-3dBh is going to result in very small numbers.

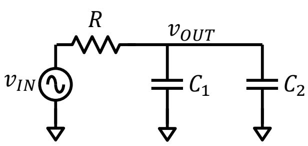
CLICK So that leaves us with an interesting first order approximation for this many pole system in the region that it starts rolling off, which is the interesting one for bandwidth estimation. Using this approximation only relies on us looking at high frequencies and on our first pole being dominant.

Sum of τ_i Equal to Sum of $OCTC_i$

- By a miracle of circuit analysis:

$$\sum_{i=0}^{n} \tau_i = \sum_{i=0}^{n} OCTC_i \text{ where } OCTC_i = C_i R_{th,i} | C_{k \neq i}\ open$$

- Dominant pole $\frac{1}{p_{-3dBh}} = \tau_{-3dBh} = \sum_{i=0}^{n} OCTC_i$

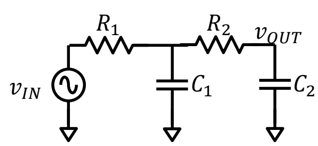


$OCTC_1 = RC_1$

$OCTC_2 = RC_2$

$\tau = RC_1 + RC_2$

$$\frac{v_{OUT}}{v_{IN}} = \frac{1}{1 + R_1(C_1 + C_2)s}$$



$OCTC_1 = R_1 C_1$

$OCTC_2 = (R_1 + R_2)C_2$

$\tau = R_1(C_1 + C_2) + R_2 C_2$

$$\frac{v_{OUT}}{v_{IN}} = \frac{1}{1 + R_1C_1s + R_1C_2s + R_2C_2s + R_1R_2C_1C_2s^2}$$

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So, great, we can have an expression for the dominant pole location, but we don't know the tau_i. And our experience with the common emitter tells us we can't find these tau_i just by Thevenizing at capacitors because poles aren't linked to a specific cap. That still leaves us in a tough spot. Fortunately, the OCTC paper got written, and it used some circuit wizardry to show that the sum of taui is equal to the sum of OCTCi, where each OCTC is a capacitance multiplied by the Thevenin resistance seen from the capacitor terminals under the condition that all other capactitors are open circuits. This second bullet just reiterates that our dominant time constant is equal to the sum of the OCTC.

CLICK, so let's take this for a test drive. Pause the video and try writing down the OCTC for this circuit and seeing if the time constant the OCTC predicts lines up with a full analysis of the time constant of the circuit.

CLICK The first OCTC for C1 is equal to R*C1. That's because C1 sees a Thevenin resistance of R to ground and C2 is open. Similarly, the OCTC for C2 is R*C2. That means we predict the dominant time constant of this system will be R1C1+R2C2. This exactly lines up with the exact transfer function I've included down here, which is a standard RC transfer function that treats C1 and C2 as if they're in parallel. Great! This works, though maybe this is a special case because the caps are in parallel or because this is first order. Let's try a trickier circuit.

CLICK Pause the video and find the OCTC for this circuit. Don't find the exact transfer function for it unless you're bored, it's a bunch of tedious algebra. I'll give you the exact transfer function to compare your OCTC to when we go over the OCTC, and I've included an appendix with a derivation of the transfer function in the notes online.

CLICK The first OCTC is given by C1*R1. We don't include R2 in this OCTC because C2 is an open circuit, so R2 is just floating when we're finding OCTC1. The second OCTC is given by R1+R2 times C2, because C1 is open so the only path the ground from the cap is through both R1 and R2. That means our overall time constant is given by the sum of these OCTC. Again, that lines up exactly with the s term of the exact transfer function I've included here. Great! We can find our first order approxiamation in a multi-pole system easily using OCTC!

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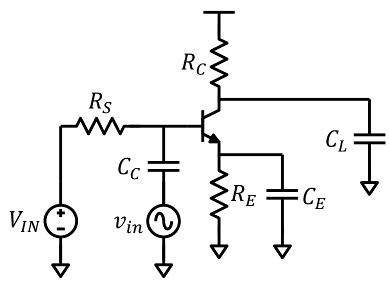
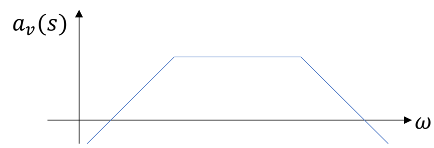
Don't Find the OCTC for Every Cap

At $s = j\omega_{-3dBh}$, these terms are 1

Recall: $a_v(s) = a_{vmb} \cdot \dfrac{s^m}{(s+p_{z1})(s+p_{z2})\ldots(s+p_{zm})} \cdot \dfrac{1}{(\tau_1 s+1)(\tau_2 s+1)\ldots(\tau_n s+1)}$



- Usually coupling, bypass cause zeros
- Usually device, load cause poles
- Shorting pole causing caps reduces a_v
- Opening zero causing caps reduces a_v



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One important note is that we don't find OCTC for every cap when we're estimating bandwidth. That's because we want to ignore caps that cause low-frequency pole-zero pairs when calculating OCTC. We expect those caps will already be shorted at w-3dBh, so we should replace them with wires when we find OCTC.

Great, but tricky in practice. How do we identify low frequency pole-zero caps? Or, alternatively, how do we identify high frequency pole causing caps? The easy way is by long experience, so when I look at a circuit I use some quick heuristics to guess which caps are intended to be shorted and which are supposed to go up to high frequency. In general, coupling caps and emitter bypass caps are intended to cause low-frequency pole-zero pairs. Device caps and load caps like the one attached to the collector cause high frequency poles.

You can also use a circuit trick to decide what caps cause high frequency poles and what caps are cause low frequency pole-zero pairs. If I imagine that I'm operating in the mid-band and gradually increasing my frequency, I'll see high frequency pole causing caps gradually become short circuits, and I'll also see the circuit's gain. So if I replace a cap with a short and it reduces my gain, then that cap has to be a high frequency pole causing cap. For instance, if I short CL, then my gain would drop to zero, so it's high frequency. I can't say the same for CC, shorting it seems like a good thing for a gain because an open CC means no signal reaches my circuit. On the other hand, if I imagine that I'm operatin gin the mid-band and gradually decreasing my frequency, I'll see low-frequency pole-zero pair causing caps become open circuits, and I'll see my gain decrease. So if I replace a cap with an open and it reduces my gain, then that cap has to be a low-frequency pole-zero pair causing cap. For instance, opening up CC reduces gain to zero, and opening up CE reduces gain from gm*RC to RC/RE.

Summary

- Predict high frequency dominant pole as
  $\frac{1}{p_{-3dBh}} = \tau_{-3dBh} = \sum_{i=0}^{n} OCTC_i$ where $OCTC_i = C_i R_{th,i} | C_{k \neq i}\ open$

- Leave zero causing caps out of analysis b/c they're already shorted
  - Usually coupling, bypass cause zeros
  - Usually device, load cause poles
  - Shorting pole causing caps reduces a_v
  - Opening zero causing caps reduces a_v

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Examples of Open Circuit Time Constants

Matthew Spencer
Harvey Mudd College
E151 – Analog Circuit Design

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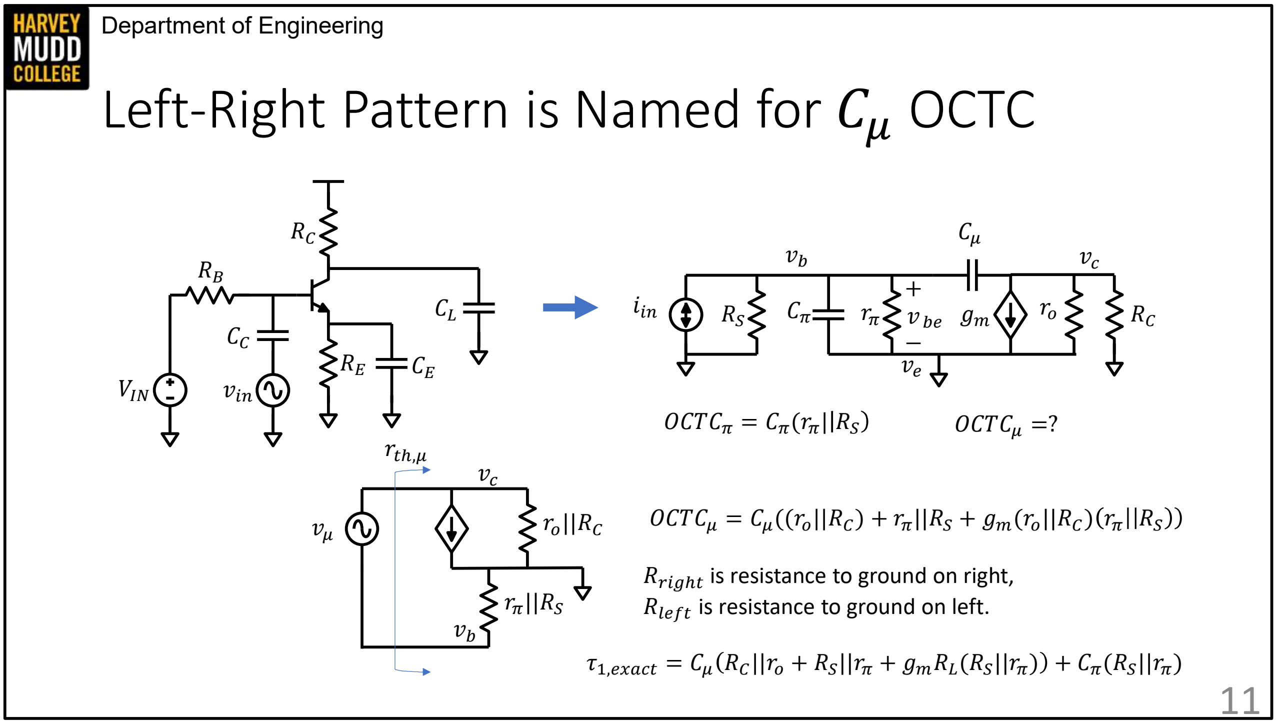
In this video we’re going to look at a few interesting examples of Open circuit time constants in action.



We’ve been wrestling with the common emitter for a while, so let’s see how OCTC handle it. I’ve drawn a common emitter on the left here to kick us off.

CLICK Here’s the dynamic small signal model of the common emitter. Note that I shorted CE and CC because they cause low-frequency zero-pole pairs, not high frequency poles. Note also that I converted vin into its Thevenin equivalent.

CLICK We can find the open circuit time constant for Cpi pretty easily. One cside of Cpi is grounded, so and the other side see rpi and Rs in parallel to ground. Because Cmu is open, we don’t have to consider any other part of the circuit because they’re not attached to Cpi.

CLICK Cmu is kind of a mess though. Neither side of Cmu is grounded, so finding the Thevenin resistance that it sees is a bit counterintuitive. Just looking at this, I don’t immediately see how to analyze Cmu.

CLICK So I’ve redrawn the circuit replacing Cmu with a test current source so that we can find rthmu. When I do this kind of redrawing, especially for OCTC, I’m really a stickler for labeling each node. Note that I’ve included vc on top, vb on bottom and ground in the middle. You often contort circuits in pretty crazy ways when you’re finding OCTCs, and I find this labeling helps me to make circuits accurately.

CLICK Fortunately, redrawing Cmu reveals a pattern that we're super used to by now: this is a left-right small signal pattern, so OCTCmu is equal to Cmu times ro||RC+rpi||RS+gm*ro||RC*rpi||RS. The left-right pattern is actually named after the behavior of Cmu. Rright is the resistance from the right side of Cmu to ground and Rleft is the resistance from the left side of Cmu to ground. You can often calculate the OCTC of caps in feedback around even complicated amplifiers like cascodes using this left-right rule, so keep an eye out for it as you analyze your first few amplifiers.

CLICK This is the s term from the exact transfer function of the common emitter, which I've copied from earlier videos. You can see that it exactly matches Cpi + Cmu, so this estimation technique looks like it does a great job for our amplifier. Notably better than the Miller approximation. On that note though, note that the Cmu OCTC includes gm*RC, so we see that Millerized caps have amplified OCTC. That's great intuition because we're often just looking for the biggest OCTC to determine how fast an amplifier will be. Even though we're not using the Miller approximation, our intuition from Miller helps us identify big OCTC.

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Worst Case Error for 2-Pole OCTC Not so Bad

$$H(s) = \frac{1}{1 + (\tau_1 + \tau_2)s + \tau_1\tau_2 s^2}$$

$$H(j\omega_{-3dB,OCTC}) = \frac{1}{1 + (\tau_1 + \tau_2)j\omega_{-3dB,OCTC} - \tau_1\tau_2\omega^2_{-3dB,OCTC}}$$

=1 by OCTC definition

Negligible if $\tau_2\omega_{-3dB,OCTC}$ is small
Worst case if $\tau_1 = \tau_2$
Therefore, $2\tau = \omega_{-3d\ ,OCTC}$
Subbing in, this term is 1/4

$$H(j\omega_{-3dB,OCTC}) = \frac{1}{1 + j - 1/4}$$

$$H_{OCTC}(j\omega_{-3dB,OCTC}) = \frac{1}{1 + j}$$

25% error low in worst case. Still conservative, good for intuition.

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However, there are times that OCTC can fail. We're going to take a look at the worst case for OCTC in a two-pole system on this slide. I've put that two pole transfer function on the top of the slide.

CLICK If we substitute the frequency we estimate by OCTC into this function, we get this expression. We know that our w-3dBOCTC is equal to one over the sum of our taui, so this first term has to be equal to 1. We said this second term was negligibly small in our earlier analysis because tau1 was a dominant pole, so tau2*w-3dB must be small. So that means the worst we could do in our estimate is if tau2 was co-located with tau1. That way neither pole dominates. If we assume that, then we know from the first term that 2*tau is w-3dBOCTC. If we sub that into this term, then we find that this term is ¼.

CLICK So our actual transfer function at w-3dBOCTC is 1/(1+j-1/4), but our OCTC approximation says the function should be 1/(1+j). That means we'll be around 25% low in our magnitude estimates for co-located poles. Errors aren't great, but this failure mode isn't so bad, which is one of the reasons OCTC are so popular. The error isn't bad, because getting within 20% with hand analysis is usually good enough to get intuition before simulating. The failure is conservative, so you get a little extra bandwidth out of your amplifiers. Finally, OCTC still tell us which caps are slowing our amplifier down, both in this case, so we can design accordingly. It's a great tool for building intuition about how different caps affect your circuit, because you know you should focus your attention on the big caps.

… (1) KNOW WHAT CAPS CAUSE PROBLEMS (2) PRETTY GOOD FOR HAND ANALYSIS

… fails conservatively.

Summary

- C_C and C_E cause zeros in common emitter amplifiers
- C_μ sees a right-left pattern with resistance to ground on right & left.
- Max OCTC error when poles are co-located. ~25% in 2 pole system.
- Even if OCTC has high error, OCTC useful:
  - Makes a conservative BW estimate,
  - Gives insight into which cap is the issue.

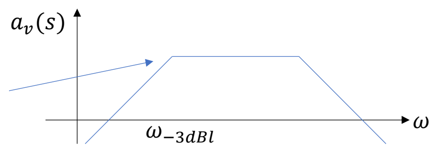
Short Circuit Time Constants

Matthew Spencer
Harvey Mudd College
E151 – Analog Circuit Design

In this video we're going to retread a slight twist on our derivation of OCTC to learn how to calculate the dominant low-frequency pole location. We'll find that location using a variant technique called Short Circuit Time Constants.

Can Estimate Lower 3dB Frequency Too

Maybe we want this too
Assume zeros at origin
and one dominant pole



$$a_v(s) = a_{vmb} \cdot \frac{s^m \cdot \left(\frac{1}{p_{z1}p_{z2} \dots p_{zm}}\right)}{\left(\frac{s}{p_{z1}}+1\right)\left(\frac{s}{p_{z2}}+1\right)\dots\left(\frac{s}{p_{zm}}+1\right)\dots} \cdot \frac{1}{\left(\frac{s}{p_1}+1\right)\left(\frac{s}{p_2}+1\right)\dots\left(\frac{s}{p_n}+1\right)}$$

$$a_v(s) = a_{vmb} \cdot \frac{s^m}{(s+p_{z1})(s+p_{z2})\dots(s+p_{zm})} \cdot \frac{1}{(\tau_1 s+1)(\tau_2 s+1)\dots(\tau_n s+1)}$$

=1 b/c all $\tau_n \omega_{-3dBl}$ small

$$a_v(s) \approx a_{vmb} \cdot \frac{s^m}{s^m + s^{m-1}(p_{z1}+p_{z2}+\dots+p_{zm}) + \dots + p_{z1}p_{z2}\dots p_{zm}}$$

$$a_v(s) \approx a_{vmb} \cdot \frac{s}{s + (p_{z1}+p_{z2}+\dots+p_{zm}) + \dots + (p_{z1}p_{z2}\dots p_{zm})/s^{m-1}}$$

≈0 b/c all $\frac{p_{zi}}{\omega_{-3dBl}}$ small for $i \geq 2$

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OK, so I've replicated our mid-band approximation here, and noted that we only really care about our dominant low frequency pole to find the low frequency -3dB point. I've also copied over the transfer function we started with.

CLICK First we do the same set of convenient transforms we did our first time through OCTC, including the weird choice of making the low frequency pole-zero pairs into canonical form.

CLICK Next, we note that our high frequency dynamics term is going to be 1 because w-3dBl is way too small to make any of the taui*w-3dB terms significant.

CLICK So we expand our denominator like before.

CLICK Then we do something tricky. We divide off s^(m-1) from both the top and the bottom of this equation. That means we have a single zero on the top, something that looks like a first order term here, and a bunch of terms that have a factor of s in their denominator.

CLICK Because we assume we have a dominant pole, each of these terms has to be small. They all include some number of pzi divided by w-3dBl, and w-3dBl is bigger than all the other pzi. That leaves us with a first order pole in the denominator, and its location is given by the sum of all the pzi.

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Sum of $SCTC_i$ Equals sum of p_{zi}

- By a miracle of circuit analysis:

  $\sum_{i=0}^{m} p_i = \sum_{i=0}^{m} 1/SCTC_i$ where $SCTC_i = C_i R_{th,i} | C_{k\neq i}\ short$

- Dominant pole $p_{-3dB} = \frac{1}{\tau_{-3dBl}} = \sum_{i=0}^{m} 1/SCTC_i$

- Open pole causing caps during this analysis b/c ω_{-3dBl} is small.

- Fancier tricks let you combine OCTC and SCTC to find 2nd order pole.

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OK, but just like OCTC, we don't know what the sum of pi is. However, circuit wizards did some analysis that shows the sum of pi is equal to the sum of inverse short circuit time constants, or SCTC, where each SCTC is a capacitance multiplied by the Thevenin impedance seen by that capacitance assuming all other caps are shorted out. The second bullet reiterates the main point: we can find our dominant low-frequency pole by adding up the inverse SCTC.

Just like OCTC, we need to ignore some caps when doing SCTC. Specifically, we need to ignore high-frequency pole-causing caps because they're all open circuits at w-3dBl.

Finally, SCTC can be combined with OCTC to locate higher order poles if you do some fancy footwork. Not worth it in my estimation because the intuition you get from this work drops off quickly.

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Summary

- Predict low frequency dominant pole as
  $p = \frac{1}{\tau} = \sum_{i=0}^{n} 1/SCTC_i$ where $SCTC_i = C_i R_{th,i} | C_{k \neq i}\ short$

- Leave pole-causing caps out of the analysis

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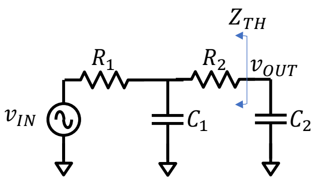
Appendix: Solving Cascaded RC Divider



$$Z_{TH} = R_2 + \frac{R_1}{R_1C_1s + 1}$$

$$v_{TH} = \frac{v_{IN}}{R_1C_1s + 1}$$

$$v_{OUT} = \frac{\frac{1}{C_2s}v_{TH}}{\frac{1}{C_2s} + Z_{TH}}$$

$$v_{OUT} = \frac{\frac{1}{C_2s}v_{TH}}{\frac{1}{C_2s} + Z_{TH}} = v_{TH}\frac{1}{1 + Z_{TH}C_2s} = v_{IN}\frac{1}{R_1C_1s + 1} \cdot \frac{1}{1 + R_2C_2s + R_1C_2s/(R_1C_1s + 1)}$$

$$\frac{v_{OUT}}{v_{IN}} = \frac{1}{1 + R_1C_1s + R_1C_2s + R_2C_2s + R_1R_2C_1C_2s^2}$$