Problem Set 1 Solutions

January 30, 2015

1

1.1

(3 points) Figure 1a is the transfer characteristic of an amplifier. If it has an input referred noise density of $NkT\Delta f$ (in units of V/Hz$^{0.5}$), what is its dynamic range? Assume it is a first-order system with a bandwidth of ω.

Solution: Let's examine the maximum and minimum signals that can be resolved at the output of this amplifier. The maximum signal is limited by signal swing, it's zero-to-peak value is $V_{o,sw}$. The minimum signal is limited by the total noise of the system. The input noise density is referred to the output by the gain squared, note that I included the system's first order roll-off in its gain.

$$v_{on}^2 = \left| \frac{V_{o,sw}/V_{i,sw}}{1 + s/\omega} \right|^2 NkT\Delta f \tag{1}$$

If we apply a quick noise integral we find

$$\bar{v}_{on}^2 = \int_0^\infty v_{on}^2 = \frac{1}{4} NkT\omega (V_{o,sw}/V_{i,sw})^2 \tag{2}$$

The dynamic range is given by the ratio of the maximum input signal – the RMS swing – to the minimum signal – the total noise voltage (which is an RMS value). Note that $\bar{v}_{on}^2$ is a measure of variance, so we need to take the square root to get $\bar{v}_{on}$ in this equation:

$$DR = \frac{V_{max,rms}}{\bar{v}_{on}} = \frac{V_{o,sw}/\sqrt{2}}{\frac{1}{2}\sqrt{NkT\omega}V_{o,sw}/V_{i,sw}} = \frac{V_{i,sw}}{\sqrt{\frac{1}{2}NkT\omega}} \tag{3}$$

Note that this result depends, as we might expect, on the ratio of the maximum input swing and the total input referred noise.

1.2

(3 points) Figure 1b is the transfer characteristic of an ADC. What is it's dynamic range?

Solution: The dynamic range is the ratio of the maximum voltage to the minimum voltage. In an ADC that is always the number of steps. Accordingly, this has a DR of 255.

1.3

(4 points) If the amplifier's output is connected to the ADCs input, what is the dynamic range of the composite system? Assume that $\sqrt{NkT\omega} = V_{i,sw}/20$.

Solution: The maximum zero-to-peak signal the ADC can accept is a code of 128, which corresponds to a swing of $4V_{o,sw}$ on the ADC input. That is much bigger than the amplifier, so the maximum zero-to-peak input swing is set to $V_{o,sw}$ by the amplifier.

The minimum signal is either the size of the ADC LSB, $V_{o,sw}/32$, or the RMS noise. Substituting the relationship given in the question in the expression for RMS noise, we find the RMS noise is $V_{o,sw}/40$, so the minimum signal is set by the ADC LSB.

$$DR = \frac{V_{o,sw}/\sqrt{2}}{V_{o,sw}/32 \cdot 1/2 \cdot 1/\sqrt{2}} = 64 \tag{4}$$

Note that it's more correct to say that the minimum signal is set by the sum of quantization noise (a concept we'll cover shortly) and input noise, but taking the maximum is the approximation I expect you to use right now. However the maximum is a bad approximation in this case since our RMS noise voltage is almost the same size as our LSB.

2

2.1

(3 points) Find the half-power bandwidth of each of the structures in Figure 2 if a signal is injected at node x and the output is measured at node y.

Solution: We will use the method of impedance to find the transfer function from x to y, then find a bandwidth by pattern matching to canonical first or second order transfer functions. Note that each of these circuits is a voltage divider. Note also that we have to use the quality factor of circuit iii to find it's badwidth: Q is defined as the half-power bandwidth divided by the natrual frequency.

$$\frac{V_{yi}}{V_{xi}} = \frac{1/Cs}{R + 1/Cs} = \frac{1}{1 + RCs} \rightarrow \omega_i = 1/RC \text{ rad/s} \tag{5}$$

$$\frac{V_{yii}}{V_{xii}} = \frac{Ls}{R + Ls} = \frac{1}{1 + Ls/R} \rightarrow \omega_{ii} = R/L \text{ rad/s} \tag{6}$$

$$\frac{V_{yiii}}{V_{xiii}} = \frac{\frac{1}{Cs}}{\frac{RLs}{R+Ls} + \frac{1}{Cs}} = \frac{\frac{R}{Cs} + \frac{L}{C}}{\frac{R}{Cs} + \frac{L}{C} + RLs} = \frac{R + Ls}{R + Ls + RCLs^2} = \frac{1 + Ls/R}{1 + Ls/R + LCs^2} \tag{7}$$

$$\rightarrow \omega_{0,iii} = \sqrt{1/LC}, \tag{8}$$

$$\omega_{0,iii}Q = R/L \rightarrow Q = R\sqrt{C/L}, \tag{9}$$

$$\frac{\omega}{\Delta\omega} = Q \rightarrow \Delta\omega_{iii} = \omega_{0,iii}/Q_{iii} = 1/RC \text{ rad/s} \tag{10}$$

1 point for each correct bandwidth.

2.2

(3 points) If a digital system were sampling node y, what would the throughput of the digital system need to be for real time processing of the samples without any loss of information? Assume the signal is "band-limited" by the bandwidth you found in the first part of this problem. For circuit iii, assume that Q is high and that our desired operating point is at resonance, so that the half power bandwidth is given by the falloff from the resonant point.

Solution:"Band-limited" should be a clue to think of the Nyquist rate. The first two circuits have a simple bandwidth from 0 to ω, so their Nyquist sampling rate are ω_i/π and ω_{ii}/π respectively – we double the frequency to honor Nyquist and then divide by 2π to convert to Hz. Sampling rates and througput are given in samples/second or Hertz, which is why we do that conversion.

There are two possible answers to the third system. One is doubling the highest frequency we care about. But recall that the $\Delta\omega$ is a little bit misleading: though it is the total bandwidth, the natural frequency isn't *aritmetically* halfway between the 3 dB cutoff frequencies. i.e.: $\omega_{hi} \neq \omega_0 + \Delta\omega$.

Instead it is geometrically bettween the upper and lower frequncy. As a result, we have to solve the following system of equations to figure out the maximum frequency:

$$\Delta\omega = \omega_{hi} - \omega_{lo} \text{ and } \omega_0 = \sqrt{\omega_{hi}\omega_{lo}} \tag{11}$$

$$\Delta\omega = \omega_{hi} - \frac{\omega_0^2}{\omega_{hi}} \tag{12}$$

$$0 = \omega_{hi}^2 - \Delta\omega\omega_{hi} - \omega_0^2 \tag{13}$$

$$\omega_{hi} = \frac{\Delta\omega}{2} + \sqrt{\frac{\Delta\omega^2}{4} + \omega_0^2} \tag{14}$$

$$\text{so } \omega_{hi,iii} = \frac{1}{2RC} + \sqrt{\frac{1}{4R^2C^2} + \frac{1}{LC}} \tag{15}$$

The frequency we need to sample at is $\omega_{hi,iii}/\pi$.

It is also possible to undersample the bandpass filter, which is the second and arguably more correct way to answer this question. It involves invoking the Bandpass Sampling theorem. If you did that, give yourself a pat on the back and assume that you're right.

Note that $2 \cdot 2\pi \cdot \omega_{0,iii}$ and $2 \cdot 2\pi \cdot \Delta\omega_{iii}$ are incorrect answers. Neither answer guarantees sampling everything in the bandwidth.

The throughput must be the same as the sample rate to process samples in real time.

1 point for each correct sampling frequency. Don't penalize yourself for the same mistake more than once: if you forgot to multiply by 2π to all three equations then only take off one point.

2.3

(4 points) For each circuit, find the total noise at node y if a noise voltage source of value $v_n^2 = NkT\Delta f$ is attached between node x and ground. Treat the resistors in circuits i and iii as "noisy" and treat the resistor in circuit ii as ideal.

Solution: For circuit i, we should model the resistor noise source as a series voltage noise of value $4kTR\Delta f$. This means the noise source attached to node x is in series with the resistor noise source, and the two can simply be added together. This means the effective input noise is $(4R+N)kT\Delta f$. This noise source is multiplied by the square of the transfer function we found in the first part of this problem and integrated:

$$\bar{v}_{n,y,i}^2 = (4R+N)kT\int_0^\infty \left|\frac{1}{1+sRC}\right|^2 df = (4R+N)kT\frac{\omega_i}{4} = \frac{(4R+N)kT}{4RC} = \left(1+\frac{N}{4R}\right)\frac{kT}{C} \tag{16}$$

Note that the contribution from the resistor itself is kT/C and is independent of R. Only increasing C can reduce the value of resistive noise. Also note that the solution is a multiple of kT/C, the multiplier in front of kT/C is abbreviated n_F (in this case $n_F = 1 + N/4R$) and is related to the noise figure of an amplifier.

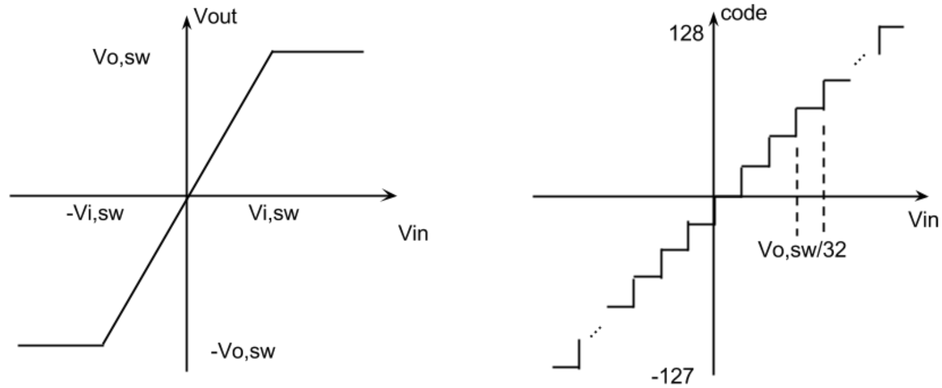
Circuit ii is also a first order system and the noise source at the input will be shaped by the transfer function in the same way as in circuit i:

$$\bar{v}_{n,y,ii}^2 = NkT\int_0^\infty \left|\frac{1}{1+Ls/R}\right|^2 df = NkT\frac{\omega_{iii}}{4} = \frac{NkTR}{4L} \tag{17}$$

We assume there is no noise in the resistor for circuit ii, so this is all the work we need to do.

For circuit iii, we should model the resistor noise as a parallel current source and find a new transfer function from that current source to the output. By superposition, node x will be grounded when we find that transfer function, which means the circuit will be a parallel RLC combination. The tranfer from the resistors noise source to the output will be the impedance of a parallel RLC tank, which we find by first finding the admittance of the tank (since parallel admittances add together) and then inverting it:

$$G_{RLC} = Cs + \frac{1}{Ls} + \frac{1}{R} \rightarrow Z_{RLC} = \frac{1}{G_{RLC}} = \frac{1}{Cs + \frac{1}{Ls} + \frac{1}{R}} = \frac{Ls}{1 + Ls/R + LCs^2} \tag{18}$$



(a) Amplifier Transfer Function (b) ADC Transfer Function

Figure 1: Transfer functions for dynamic range calculation

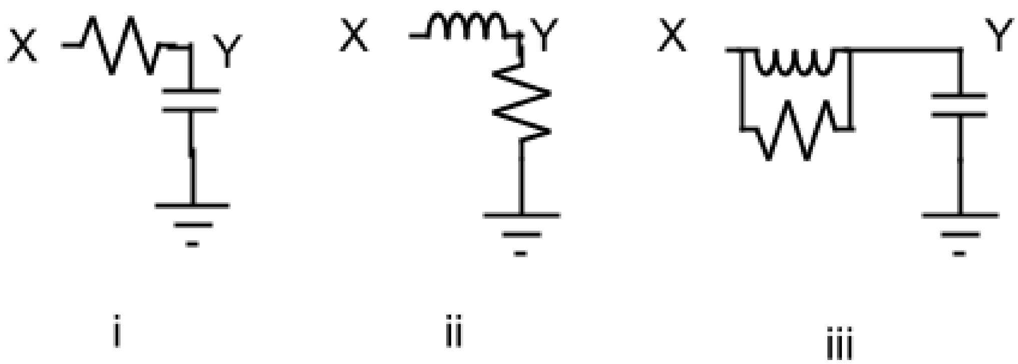


Figure 2: Noise and Bandwidth Circuits.

The sum of the noise densities at the output is as follows:

$$\bar{v}_{y,n}^2 = \int_0^\infty v_n^2 \left|\frac{V_y}{V_x}(s)\right|^2 + i_r^2 |Z_{RLC}(s)|^2 = \int_0^\infty NkT \left|\frac{1 + Ls/R}{1 + Ls/R + LCs^2}\right|^2 \Delta f + \frac{4kT}{R}\left|\frac{Ls}{1 + Ls/R + LCs^2}\right|^2 \Delta f \tag{19}$$

Terrible pattern matching, factoring and substitution follows to give the result

$$\bar{v}_{y,n}^2 = kT\left[\frac{N\omega_0 Q}{4}\left(1 + \frac{\omega_0^2}{\omega_z^2}\right) + \frac{4L}{RC} \cdot \frac{\omega_0 Q}{4}\right] \text{ Note we need to multiplyby } \omega_z/\omega_0 \text{ in the second term} \tag{20}$$

$$= kT\left[\frac{NR}{4L} + \frac{NR}{4L} \cdot \frac{L^2/R^2}{LC} + \frac{4L}{RC} \cdot \frac{R}{4L}\right] \tag{21}$$

$$= \frac{kT}{C}\left(\frac{NRC}{4L} + \frac{N}{4R} + 1\right) \tag{22}$$

1 point for correct answer for circuit i, 1 point for correct answer for circuit ii. For circuit iii, 1 point for noise integral setup and 1 point for evaluation.