Lecture 05: Smith Charts

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Generalized Reflection Coefficient

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In this video we're going to find the reflection coefficient of a new load, which is a finite transmission line terminated in Zl

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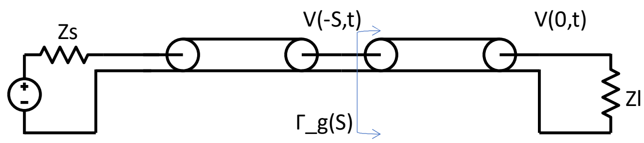
Transmission Lines Change Reflection Coeff.



$\Gamma = \Gamma_g(0) = \dfrac{V_-(0,t)}{V_+(0,t)}$ — Definition of reflection coefficient & sub in wave solution

$\Gamma_g(S) = \dfrac{V_-(0,t)e^{\gamma S}}{V_+(0,t)e^{-\gamma S}} = \Gamma e^{2\gamma S}$ — Use propagation to add phase back to –S, then relate to Γ

$\Gamma_g(S) = \Gamma e^{2jkS}$ — If lossless

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This slide includes a picture of this new load, which we'll call a delayed load. We're going to calculate the reflection coefficient looking into this delayed load, so CLICK we're going to calculate Gamma at this spot on a transmission line. We're going to call this Gamma the generalized reflection coefficient and give it the symbol Gamma_g.

We're going to find that Gamma_g depends on S, the length of the transmission line on the delayed load. That fact suggests the driving point impedance of this load changes as S changes, which has applications for circuit design. This fact allows us to impedances that would difficult to build as lumped circuits. So we're motivated by using transmission lines to make interesting looking load impedances. Put a pin in that for now, we'll talk more about how to calculate the impedance of delayed loads soon.

In the mean time, we know that Gamma_g(S) is going to be V+ of (-S, t) divided by V- of (-S,t) , but using that as the starting point for our derivation will lead us in circles because we usually calculate V- using Gamma, which we don't know yet. Instead, we're going to figure out our regular old Gamma at Zl, which is Gamma_g of 0, and then we'll propagate the signals we used to find Gamma backwards along the transmission line.

CLICK We've started that strategy here by remembering the definition of Gamma. There are two equations on this line, and the first on is just pointing out the cute fact that regular old Gamma is the generalized Gamma with a zero-length transmission line. The second relation reminds us that Gamma is the ratio of the left-travelling wave to the right-travelling wave at the termination.

CLICK We can then make a quick leap to a formula for the generalized reflection coefficient. We know that the voltage at location S is going to be given by scaling our voltage at zero by e to the propagation constant. The right-travelling wave loses phase when we travel backwards by S, while the left-travelling wave gains phase travelling backward, so the propagation constant has different signs on the top and bottom of this expression. We get the final expression on this line by recognizing two things. First, the ratio of V- of 0,t and V+ of 0,t still appears in this equation, and it's still equal to Gamma. Second, the phases of the two exponentials add together to double the propagation constant.

CLICK We'll be looking at lossless lines a lot of the time, and if we do that gamma is equal to j times the wave number and Gamma_g is revealed to just be a phase-shifted version of Gamma. You might recognize this expression: a similar term appeared when we were deriving our voltage standing wave pattern. That's because the changes in impedance we see looking into a generalized load are caused by voltage standing wave patterns. We'll explore that very soon.

Summary

- Delayed loads have transmission lines attached to them. They can make interesting impedances.
- The generalized reflection coefficient is a phase-shifted version of the load reflection coefficient (in a lossless line).

$$\Gamma_g(S) = \Gamma e^{2jkS}$$

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Driving Point Impedance Transforms for Sinusoids

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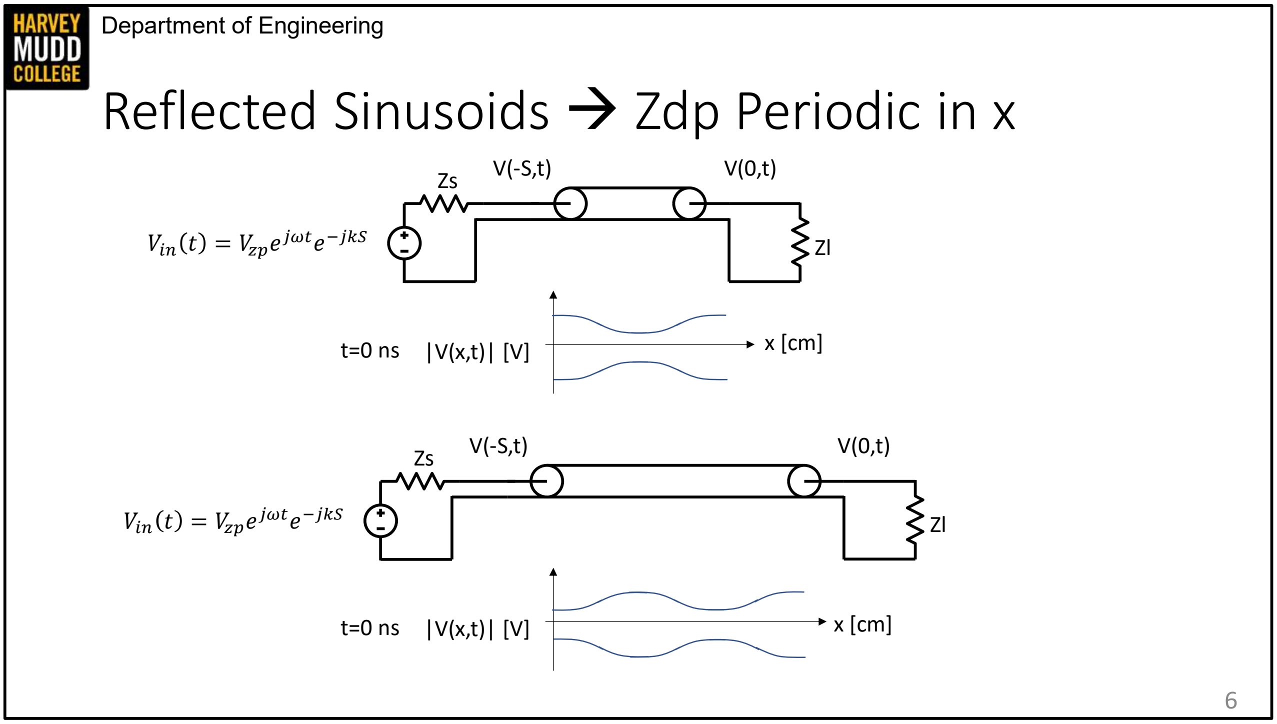
In this video we're going to try to figure out how the driving point impedance of transmission lines changes when we add different lengths of transmission line to a load.



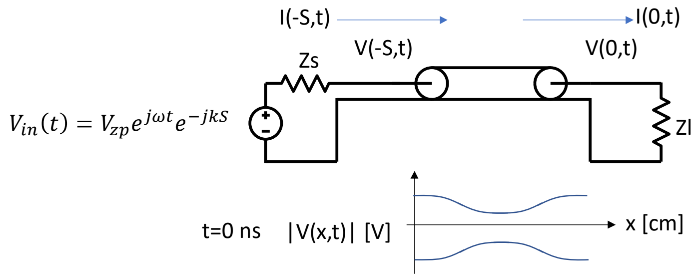
We've already looked at how the reflection coefficient changes when we add transmission lines to a load, and this video is tightly related to that idea. Here we're looking at how impedance changes with lengths of transmission line, which is what causes our generalized Gamma to change as we add lengths of a transmission line to a load.

The voltage standing wave pattern is what causes of changes in impedance with transmission line length. I've drawn two transmission lines with standing wave patterns on them to show that. The driving point impedance is set by the ratio of V(-S,t)/I(-S,t) on this line, which is equivalent to saying that the driving point impedance is the impedance such that the voltage divider Zdp/(Zs+Zsp) results in the voltage at V(-S,t). That second definition is handy when we're thinking about Zdp for a sinusoidal wave on a finite line, because we know the amplitude of waves is going to be set by the voltage standing wave pattern, and so we pick a Zdp that divides us down to the standing wave pattern.

Because the standing wave pattern is periodic, Zdp is going to vary periodically as a function of the length of the transmission line. You can see a suggestion of that on this slide, the driver sees a antinode in the voltage standing wave pattern on the top transmission line, which means Zdp needs to be big. The bottom line contains more of the standing wave pattern such that the driver sees a node, which means Zdp needs to be small. If the line length were somewhere between these two, you would tap into a different spot in the standing wave pattern, which would require a different Zdp. Or if the lines were exactly one wavelength longer you'd be tapping into the same spot in the pattern, implying the same Zdp.

That's a little complicated, but there's a silver lining. Even though the standing wave pattern makes Zdp change with length, the fact the pattern is a standing wave means Zdp doesn't change with time. Also, though these pictures are great, we need a mathematical model to let us do more interesting calculations.

Find Zdp(S) By Propagating Back From Zdp(0)



$$Z_{dp}(0) = \frac{V(0,t)}{I(0,t)} = \frac{V_+(0,t) + V_-(0,t)}{I_+(0,t) - I_-(0,t)} = \frac{V_+(0,t)}{I_+(0,t)} \frac{1 + \frac{V_-(0,t)}{V_+(0,t)}}{1 - \frac{I_-(0,t)}{I_+(0,t)}} = \mathrm{Z}_0 \frac{1 + \Gamma_{\mathrm{g}}(0)}{1 - \Gamma_{\mathrm{g}}(0)}$$

$$Z_{dp}(S) = \mathrm{Z}_0 \frac{1 + \Gamma_{\mathrm{g}}(-\mathrm{S})}{1 - \Gamma_{\mathrm{g}}(-\mathrm{S})} = \frac{1 + \Gamma e^{2\gamma S}}{1 - \Gamma e^{2\gamma S}}$$

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I've brought over an image of the transmission line to help us start making that mathematical model. I've also added some arrows indicating the current flow at the source and the load.

We know that Zdp(S) is given by V(-S,t)/I(-S,t), but that's a tough place to start this calculation because we don't know the voltage and current at -S. Instead we're going to use the same approach we used on the generalized reflection coefficient: we'll calculate the voltage and current at the load, and then propagate those back to the driver to find the ratio there.

CLICK our first set of equations is all about rearranging the voltage and current at the load. We know Zdp of S=0 is given by V of (0,t) over I of (0,t), and we rewrite those as left and right travelling waves. We can factor V+ out of the top of the equation and I+ out of the bottom, to get the equation in some handy forms. The V+ over I+ ratio in front becomes Z0, while the V- over V+ and I-over I+ ratios are each given by the reflection coefficient, which I've chosen to write as the generalized reflection coefficient at zero.

CLICK That form suggests that finding Zdp at point S is as simple as substituting our expressions for generalized Gamma into the Zdp of S=0 equation. We do that in this second set of equations and wind up with a tidy looking expression for Zdp.

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Terrible Math Reveals and Interesting Ratio

$$Z_{dp}(S) = \frac{1+\Gamma e^{2jkS}}{1-\Gamma e^{2jkS}}$$

From last page

$$\frac{Z_{dp}(S)}{Z_0} = \frac{\frac{Z_L}{Z_0}(e^{-\gamma S}+e^{\gamma S})+(e^{-\gamma S}-e^{\gamma S})}{\frac{Z_L}{Z_0}(e^{-\gamma S}-e^{\gamma S})+(e^{-\gamma S}+e^{\gamma S})}$$

Sub in Γ=(Zl-Z0)/(Zl+Z0), terrible algebra

$$\frac{Z_{dp}(S)}{Z_0} = \frac{\frac{Z_L}{Z_0}-\tanh(\gamma S)}{1-\frac{Z_L}{Z_0}\tanh(\gamma S)}$$

Combine exponentials, awkward

$$\frac{Z_{dp}(S)}{Z_0} = \frac{\frac{Z_L}{Z_0}-j\tan(kS)}{1-j\frac{Z_L}{Z_0}\tan(kS)} = \frac{Z_L\cos(kS)-jZ_0\sin(kS)}{Z_0\cos(kS)-jZ_L\sin(kS)}$$

If lossless, still awkward

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We can simplify this expression further, and we're going to do so on this page. CLICK first, we're going to skip a lot of algebra in order to rearrange this expression in terms of Zdp/Z0, which we can call our normalized driving point impedance. CLICK second we're going to combine complex exponentials into hyperbolic tangents, which gives us a general expression for the normalized Zdp. One of the cool things about this derivation is that it works for lossy lines, and you'll notice that this expression depends on gamma rather than just the wave number. CLICK However, the expression for lossless lines is somewhat easier to deal with. You can see from the combination of sines and cosines that we'll be tracing out some kind of circle in the Z plane as S is increased.

This Formula Has Interesting Implications

$$\frac{Z_{dp}(S)}{Z_0} = \frac{\frac{Z_L}{Z_0} - j\tan(kS)}{1 - j\frac{Z_L}{Z_0}\tan(kS)}$$

- Zl=Z0 means Zdp is always Z0, independent of S.
- ¼ wavelength on the line means kS= π/2 → inverse impedance?
- ½ wavelength on the line means kS= π → tan is periodic in π

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This formula has some wild implications. The first, and most commonly used, is that matched loads always have a driving point impedance of Z0, regardless of what S is. This is really useful when interconnecting RF systems, it means that the cabling or routing won't affect your performance. RF systems are often designed to be matched loads in order to take advantage of this.

Second, if you have a quarter of a wavelength on your line, then the tangent of kS is infinity and we find that Zdp = Z0^2 over Zl. So, by adding some transmission line, we have inverted our impedance. That's weird for resistors. That's really weird for open circuits, which become shorts. That's really, really weird for capacitors, which start to look like inductors. Those are all fun party tricks!

Finally, tangent is periodic in pi, so every additional half-wavelength we add to our transmission line gives us the same Zdp.

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Summary

- Driving point impedance of a line varies periodically as a function of the length of the line according to a tricky equation:

$$\frac{Z_{dp}(S)}{Z_0} = \frac{\frac{Z_L}{Z_0} - j\tan(kS)}{1 - j\frac{Z_L}{Z_0}\tan(kS)}$$

- The driving point impedance of terminated lines doesn't change with length.

- This driving point impedance transform is capable of making some strange impedances.

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The Smith Chart

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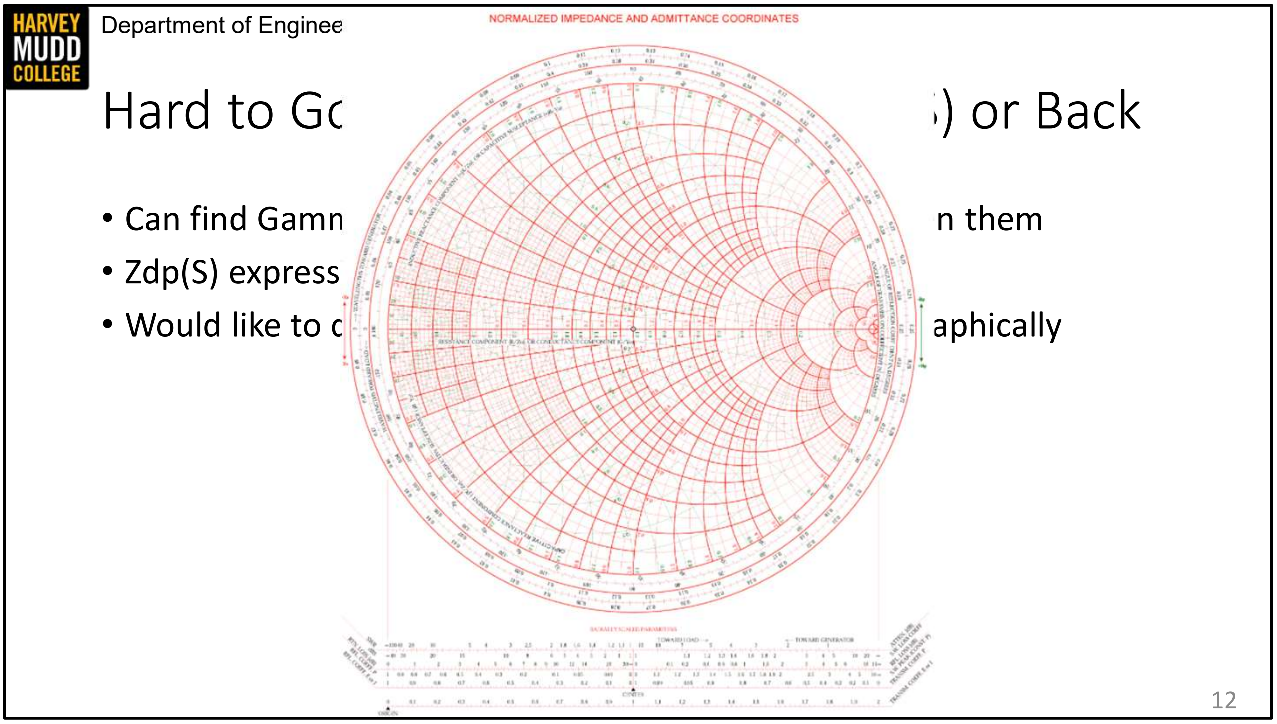
In this video we're going to learn about a graphical tool that helps us relate reflection coefficients, generalized reflection coefficients, load impedances and driving point impedances.



OK, so we have a problem. We can calculate the generalized reflection coefficient pretty easily, and we can turn the crank on our driving point impedance formula if we have to, but these tools aren't great for design. We'd really like to have a fast way to switch back and forth between Gamma and perceived impedance, either a Zl or a Zdp depending on the load. In a perfect world, this technique would be quick enough that we can apply it to measured data in our head. That points to using graphical techniques because those don't require us to do too many calcuations.

It turns out that this technique exists, and it's one of the most intimidating graphs in engineering. You might say, "Prof. Spencer, I'm not intimidated by graphs. I've seen it all at this point. Heck, you've already showed me Bode plots." Well feast your eyes on the Smith Chart! CLICK

This does look bad, but it's actually pretty reasonable. This is a plot of the complex Gamma plane with resistance and reactance coordinates laid on top of it. We'll figure it out in the next few slides.

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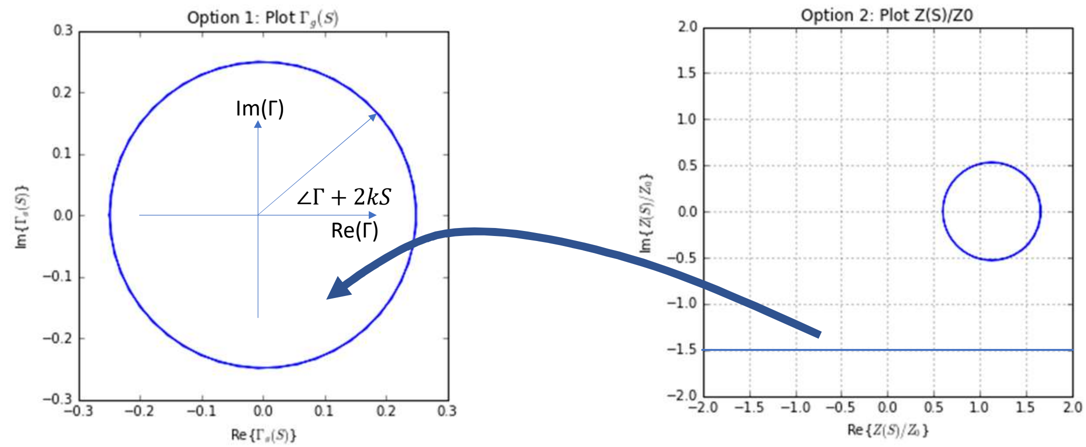
It’s Easier to Plot Γ than Z



Pro: Can use phasors

Con: Open circuit is infinite

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We’re going to take a step back to build up to the full Smith Chart. We know we’re making a graphical tool to relate Gamma and Z, so the first thing we need to decide is whether we’re plotting Gamma or Z. I’ve plotted both on this page as functions of S to help us make our decision. The main thing to point out is that Gamma_g(S) describes a circle in the complex plane, it’s just a vector rotating around in the Gamma plane, so it becomes a nice phasor.

The plot of Z(S) isn’t too badly behaved either, it just turns into a circle somewhere else in the complex plane as S moves around. However, there’s a bigger problem with plotting Z: which is that sometimes we use open circuits. So we’d have to figure out how to plot infinite values if we wanted to use a graph of Z

Both of these facts suggest we should build our tool by making a graph of the complex Gamma plane, and indeed that’s what the Smith Chart is. In order to switch between Gamma and Z, we’re going to take the grid lines on the Z plot and superimpose them on the Gamma plane. So CLICK we need to figure out what this line looks like when we put it on the left graph.

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Z Lines Become Circles on the Γ Plane

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{\frac{Z_L}{Z_0} - 1}{\frac{Z_L}{Z_0} + 1} = \frac{Z_n - 1}{Z_n + 1}$$

This is a Mobius Transformation. Turns circles to circles. Lines are infinite circles

$$\mathrm{Re}\{\Gamma\} + j\mathrm{Im}\{\Gamma\} = \frac{(R_n + jX_n) - 1}{(R_n + jX_n) + 1}$$

Substitute in complex representations. Separate into real and imaginary parts.

$$\left(\mathrm{Re}\{\Gamma\} - \frac{R_n}{1 + R_n}\right)^2 + \mathrm{Im}\{\Gamma\}^2 = \frac{1}{(1 + R_n)^2}$$

Then rearrange ad nauseum to find loci of constant Rn and Xn.

$$(\mathrm{Re}\{\Gamma\} - 1)^2 + \left(\mathrm{Im}\{\Gamma\} - \frac{1}{X_n}\right)^2 = \frac{1}{X_n^2}$$

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We're going to map those lines from Z to Gamma using a slight modification of our standard equation for Gamma. By factoring Z0 out of the top and bottom, we get it in normalized form where Zn, which is equal to Zl/Z0, is called the normalized impedance. This form is an example of a mathematical function called a Mobius transformation. Mobius transformations have a special property, which is that they preserve circles, so anything that would describe a circle in Z space will become a circle in Gamma space. This is true for very inclusive definitions of circles, most notably it includes lines, which can be interpreted as circles that pass through infinity.

We're going to stretch this expression to the breaking point on this slide, but all of the manipulations are just high school algebra on the real and imaginary parts of it. CLICK So we get started by substituting Rn plus jXn for Zn, where Rn is normalized load resistance and Xn is normalized load reactance. We also split Gamma into its real and imaginary parts. Once we've done that, we have actually stealthily converted this equation into two equations: we know the real parts of the right and left side have to be equal and so do the imaginary parts. We could rearrange this equation to split up the real and imaginary parts, but I'm not going to do the math here. Instead, we're just going to cut to the result CLICK. The derivation that gets you to these equations is actually pretty cool, but I don't have time for it in this video. I recommend you look through the derivation I've linked on the course site. As enticement, it includes the only use of completing the square that I've seen out in the wild.

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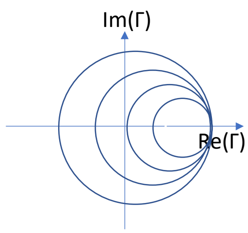
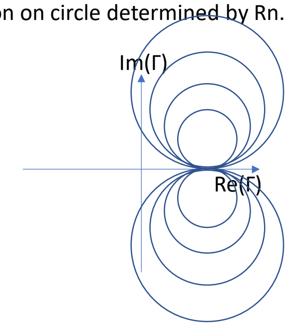
Overlaying Loci on the Gamma Plane

$$\left(\mathrm{Re}\{\Gamma\} - \frac{R_n}{1+R_n}\right)^2 + \mathrm{Im}\{\Gamma\}^2 = \frac{1}{(1+R_n)^2} \qquad (\mathrm{Re}\{\Gamma\} - 1)^2 + \left(\mathrm{Im}\{\Gamma\} - \frac{1}{X_n}\right)^2 = \frac{1}{X_n^2}$$

Specify Rn → a circle on the Γ plane
center = (-Rn/(1+Rn) , 0)
radius =1/(1+Rn).
Position on circle determined by Xn.



Specify Xn → a circle on the Γ plane
center = (1 , 1/Xn)
radius = 1/Xn.
Position on circle determined by Rn.



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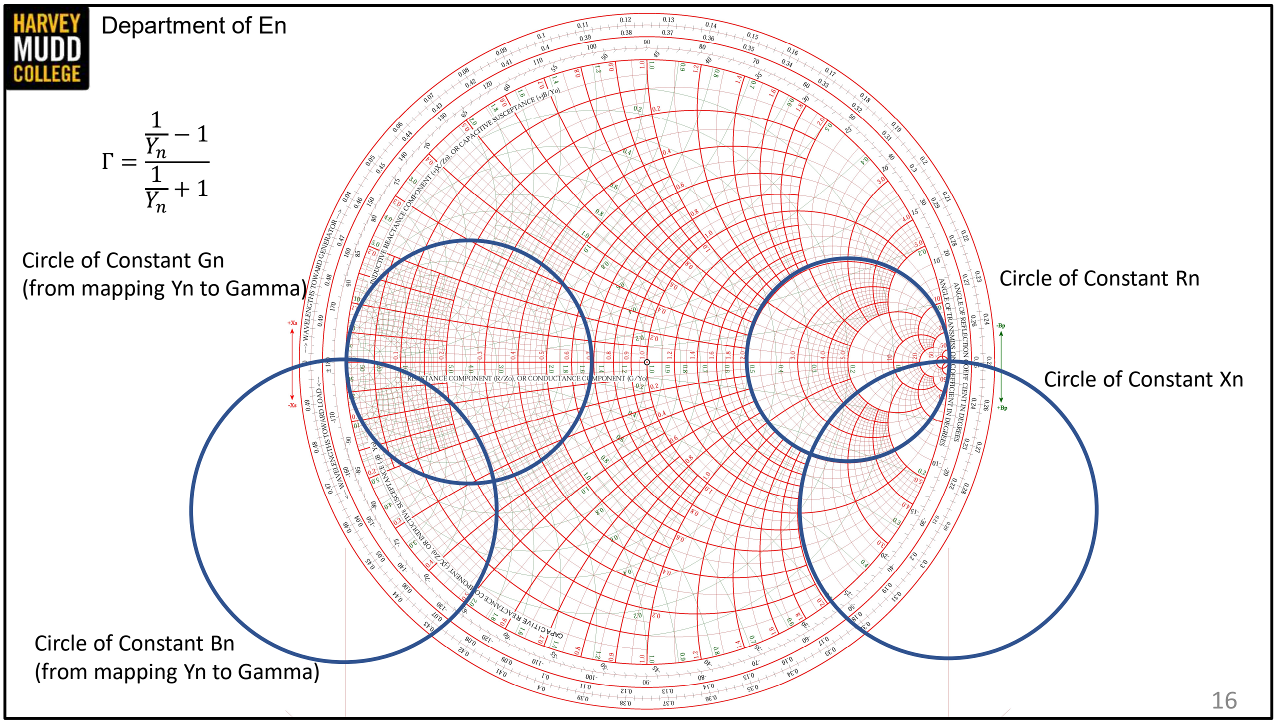
A quick glance at this result is promising. In the Z plane, we had vertical grid lines with constant resistance and variable reactance, and our first equation describes a locus of points for a constant resistance and variable reactance. The Z plane also had horizontal grid lines with constant reactance values and variable resistance, and our second equation describes a locus in the complex plane with constant reactance and variable resistance.

Both of the loci specified by these equations are circles. The constant resistance circles have a center at Rn over 1 plus Rn and a radius of 1 over 1 plus Rn. If you add the center and the radius together, you can see the point 1,0 is going to be included on every circle regardless of what Rn is. I've drawn a picture of that below. The constant reactance circles have some similar behavior. They have a center with a real value of 1 and a complex value of1 over Xn, and the radius is 1/Xn. Subtracting the radius from the center shows that 1,0 will be a part of this family of circles too, regardless of whether Xn is positive or negative. I've drawn that below too.

These sets of circles make up our resistance and reactance grid on the Gamma plane. If we overlay them, we'll get a Smith Chart.



We can see that here. CLICK This circle is an example of constant resistance, and CLICK this circle is an example of constant reactance. There's also shadowy lines on this Smith Chart that look like a reversed version of the Smith Chart. Those come from the same derivation we opted to write Gamma in terms of admittance instead of impedance. The equation at the root of that parallel derivation looks like this CLICK. It gives rise to circles of constant conductance like this CLICK and circles of constant susceptance like this CLICK.

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Summary

- The Smith Chart is a tool to convert between Γ and Z (or Y)
- It’s a plot of normalized Z (or Y) coordinates on the Γ plane
- The axes are made of circles of constant Rn and constant Xn
- You can flip it around to make a shadow Smith Chart for Gn and Bn

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Smith Chart Examples

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In this video we're going to see a few examples of Smith Charts to get some practice with them.

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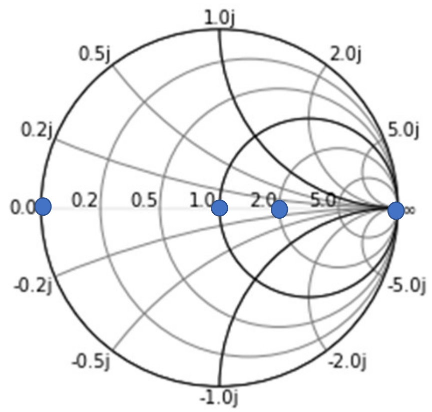
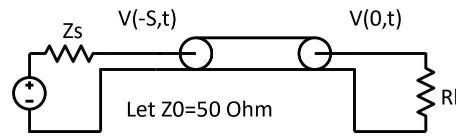
Real Zl is on the Real Axis of Smith Chart





Point	Rl	Γ	Rn	Xn
1	50	0	1	0j
2	100	1/3	2	0j
3	open	1	Infinite	0j
4	Short	-1	0	0j

- Note: Point 1 is on the Rn=1 circle, on the Xn=0j circle.
- Outer ring is |Γ|=1

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We're going to start with some familiar resistive terminations. We're going to assume Z0 is a purely resistive 50 ohms in all of these examples.

The table on this page summarizes some common terminations, and I've drawn point 1 on the Smith Chart on the left already. When we terminate a 50 ohm line in 50 ohms, we expect there to be no reflections. That means both the real and complex part of Gamma are zero, so our point is at the origin of the Gamma plane. It also means that Rn is equal to 1 and Xn is equal to zero – as a reminder, we found those by dividing the real and imaginary parts of Zl by 50 ohms. We can see that point 1 is on the Rn=1 circle and the Xn=0 circle, which is just the x-axis of the Gamma plane.

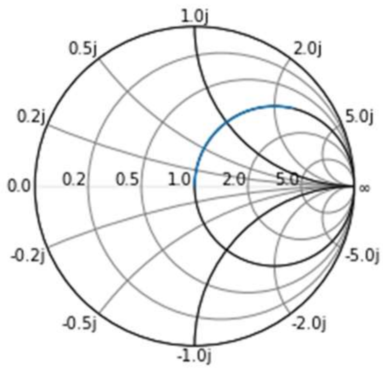
If we double the load resistance, Gamma becomes 1/3 plus 0j, and the normalized impedance becomes 2+0j. Take a second and think about where you'd put that point on this plot. CLICK This is the right spot. We can certainly see that it's on the Rn=2 circle and the Xn=0j circle. It's harder to tell that it's at Gamma = 1/3 + 0j, but it's plausible. We'd be able to judge exactly what Gamma value it was more easily if we knew the radius of this outermost circle we keep drawing.

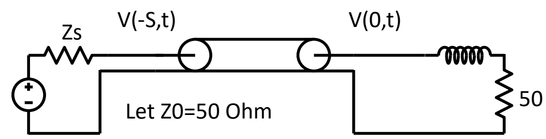
As a way to figure that out, we could imagine drawing point 3, which changes Rl to an open circuit. That results in Gamma=1, Rn=infinity and Xn still being 0j. Think about where you'd put point 3. CLICK There's no Rn=infinity circle drawn on here, but we can see that Rn circles get smaller as we increase Rn (and you may recall the radius of them was 1/(1+Rn)), and we know that all the constant resistance circles intersect at (1,0), so point 3 winds up all the way on the right side of the circle, which we've now identified as having a radius of 1. To say that another way, the outer circle is the unit circle in the Gamma plane.

Point 4 represents a short load, so Gamma is -1 and Rn is zero. Where do you think that point will be? CLICK That point winds up all the way on the left of this circle. Points 3 and 4 make point 2 look pretty reasonable, it's about a third of the way to the outer circle.

Real load impedances result in real Gamma values, and we can see that all of these points fell on the real axis of the Gamma plane.

Series Inductor Moves Along Constant Xn Arc



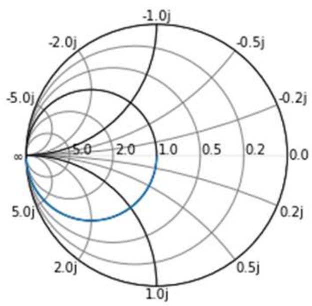


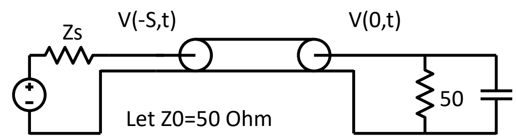
- On the Rn=1 circle, the Xn increases with ω.
- Capacitors have negative Xn because 1/j = -j

The Smith Chart becomes really useful when we start making tricky loads like this one. Here we have an inductor in series with a 50 ohm resistor. Because the impedance of an inductor is jwL, we're going to see this load trace out a locus of points on the complex plane as we vary w. For very low w, we know the inductor looks like a short, so we'd expect the locus to start at 0,0 on the complex plane in honor of Rl matching Z0. For high w the inductor looks like an open, so we'd expect the locus to finish up at 1,0 on the complex plane. In between, we'd move around the constant reactance circle for Rn=1 as our value of Xn becomes more and more positive. We see exactly that on the smith chart: a locus that includes 0,0 and that climbs across different Xn values along the Rn=1 circle.

Series Inductors move around the positive half-plane like this because jwL is a positive reactance. Series Capacitors would move around the negative half-plane because 1 over jwC is equal to –j/wC, if we multiply top and bottom by j, which is a negative reactance.

Shunt Capacitor Moves Along Constant Bn Arc





- On admittance axes, positive jBn maps to negative complex Gamma

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Another tricky termination would be a shunt capacitor. At low frequencies it would be open, so we'd expect a reflection coefficient of 0 at the middle of the Smith Chart. At high frequencies it will be a short circuit, so showing up at the Gamma=-1 point on the left of the Smith Chart. Describing what happens in between demands that we use the Admittance and Suscpetance lines instead of the Reactance and Resistance Lines. Our load conductance Gl is 1/50 S, and our characteristic conductance is also 1/50 Siemen, so the normalized conductance is 1. Our normalized susceptance is jwC over 1/50, so we'll sweep out a path on the complex plane as w is changed. One oddity of the admittance axes is that positive jBn on the Y plane maps to circles on the negative Imaginary Gamma half-plane, so this curve moves across the positive jBn coordinates which are on the bottom half of the Smith Chart. That's a little weird to remember, but there's a convenient pneumonic: the bottom of the Smith Chart is for capacitors (either in shunt or series) and the top half is for inductors (either in shunt or series).

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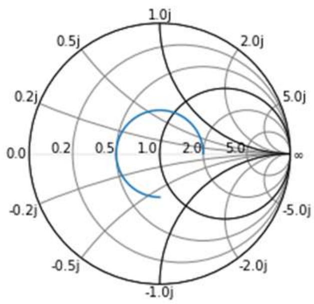
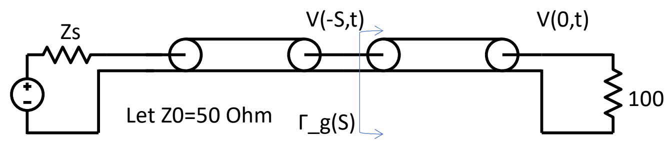
Load T Line Rotates Reflection Coefficient





- Just the same as the Γ_g(S) equation, why we chose the Γ plane.

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Finally, if we look at our delayed load we can see that the locus of points it occupies as frequency changes is described by a circle that's concentric with the unit circle. This is just our generalized Gamma_g being swept around the plane as the phase it sees increases. We picked this representation because it would be easy to draw Gamma_g as a phasor, and we're seeing that here.

Summary

- Smith Charts let us translate easily between loads and reflection coefficients.
- Inductors and capacitors trace out loci of different reflection coefficients as frequency is varied. C on bottom ½ plane, L on top.
- Admittance axes map positive jBn to negative j*IM{Γ}