The Black-Scholes Model

... pricing options and calculating Greeks

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Conceptually calculating what a 110 OTM call option should be worth if the present price of the stock is 100 ...
How Black-Scholes works ...

The Black-Scholes model is used to price European options (which assumes that they must be held to expiration) and related custom derivatives. It takes into account that you have the option of investing in an asset earning the risk-free interest rate.

It acknowledges that the option price is purely a function of the volatility of the stock's price (the higher the volatility the higher the premium on the option).

Black-Scholes treats a call option as a forward contract to deliver stock at a contractual price, which is, of course, the strike price.

The Essence of the Black-Scholes Approach

- Only volatility matters, the \( \mu \) (drift) is not important.
- The option's premium will suffer from time decay as we approach expiration (Theta in the European model).
- The stock's underlying volatility contributes to the option's premium (Vega).
- The sensitivity of the option to a change in the stock's value (Delta) and the rate of that sensitivity (Gamma) is important [these variables are represented mathematically in the Black-Scholes DE, next lecture].
- Option values arise from arbitrage opportunities in a world where you have a risk-free choice.
The Black-Scholes Model: European Options

\[ C = SN(d_1) - Ke^{-r(\sqrt{365})N(d_2)} \]

- \( C \) = theoretical call value
- \( S \) = current stock price
- \( N \) = cumulative standard normal probability dist.
- \( t \) = days until expiration
- \( K \) = option strike price
- \( r \) = risk free interest rate
- \( \sigma \) = daily stock volatility

**Note:** Hull's version (13.20) uses annual volatility. Note the difference.

Breaking this down ...

\[ C = SN(d_1) - Ke^{-r(\sqrt{365})N(d_2)} \]

This term discounts the price of the stock at which you will have the right to buy it (the strike price) back to its present value using the risk-free interest rate. Let's assume in the next slide that \( r = 0 \).

\[ d_1 = \frac{\ln(S/K) + (r/365 + \sigma^2/2)t}{\sigma\sqrt{t}} \]

Dividing by this term (the standard deviation of stock's daily volatility adjusted for time) turns the distribution into a standard normal distribution with a standard deviation of 1.
... or simplifying it some

\[ CP = SP \times \Pi(d_1) - STR \times \Pi(d_2) \]

This is the absolute log growth difference between the strike price and the stock price.

\[ d_1 = \frac{\ln\left(\frac{SP}{STR}\right) - \sigma^2/2}{\sigma} \]

We are calculating the cumulative probability to this standard normal point.

\[ d_2 = \frac{\ln\left(\frac{SP}{STR}\right) - \sigma^2/2}{\sigma} \]

This normalizes it to standard normal (the numerator is now “number of standard deviations.”

\( \mu \) is zero so this is the log-normal zero mean adjustment.

... assume that \( r \) is 0 and \( t \) is 1:

\[ C = S \times N(d_1) - K \times N(d_2) \] (assuming \( r \) to be 0)

... and some more

This term, our \( x \) of two slides ago, represents the spread in continuous growth terms between the stock price and the strike price, and when normalized by the denominator, the spread as the number of standard deviations. For example, if \( S = 110 \) and \( K = 100 \) and volatility = 10%, then this terms equals 9.5%, or about one standard deviation. \( x > 0 \) for itm calls and otm puts and \( x < 0 \) for otm calls and itm puts.

\[ d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(\frac{r}{365} + \sigma^2/2\right)t}{\sigma\sqrt{t}} \]

This term has the effect of removing the bias.

\[ d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(\frac{r}{365} - \sigma^2/2\right)t}{\sigma\sqrt{t}} \]
Using the Black-Scholes Model

There are variations of the Black-Scholes model that prices for dividend payments (within the option period). See Hull section 13.12 to see how that is done (easy to understand). However, because of what is said below, you really can’t use Black-Scholes to estimate values of options for dividend-paying American stocks.

There is no easy estimator for American options prices, but as Hull points out in chapter 9 section 9.5, with the exception of exercising a call option just prior to an ex-dividend date, "it is never optimal to exercise an American call option on a non-dividend paying stock before the expiration date."

The Black-Scholes model can be used to estimate "implied volatility". To do this, however, given an actual option value, you have to iterate to find the volatility solution (see Hull’s discussion of this in 13.12). This procedure is easy to program and not very time-consuming in even an Excel version of the model.

For those of you interest in another elegant implied volatility model, see Hull’s discussion of the IVF model in 26.3. There you will see a role played by delta and vega, but again you would have to iterate to get the value of the sensitivity of the call to the strike price.

Calculating implied volatility with B/S:

\[
d_1 = \frac{\ln\left(\frac{SP}{STR}\right) + \sigma^2}{\sigma} \div 2
\]

Very easy to do: Once Black-Scholes is structured, you can use an iterative technique to solve for \( \sigma \).
VBasic iterative technique used in IDV master

Below is the actual calculation of implied volatility.
The Ringer is for testing temporary values in construction only.

Do
CIPD = CIPD + 0.00001
DeNom = Log(StockPR / StrikePR) + ((IntRR / 365) + (CIPD ^ 2) / 2) * DTMR
DurVol = CIPD * DTMR ^ 0.5
DND1 = WorksheetFunction.NormSDist(DeNom / DurVol)
DND2 = WorksheetFunction.NormSDist(DeNom / DurVol - DurVol)
Ringer = Exp(-IntRR * DTMR / 365)
TempCallPR = StockPR * DND1 - StrikePR * Exp(-IntRR * DTMR / 365) * DND2

Loop Until TempCallPR >= CallPR

Command below writes a value back to a named designated cell
Range("CIPD").Value = CIPD

Calculating IDV for strangles (V. 3.3)

Name: Gary R. Evans
Date: March 30, 2012

One Year:
Average DGR: 0.00038
Standard Deviation: 0.01299
Average ABS DCGR: 0.00909

60 day:
Average DGR: 0.00109
Standard Deviation: 0.00565
Average ABS DCGR: 0.00415

Example: March 30, 2012 weekend strangle
An example ...

Consider an itm option with 20 days to expiration. The strike price is 105 and the price of the stock is 100 and the stock has an daily volatility of 0.02. Assume an interest rate of 0.01 (1% annual).

\[
d_1 = \frac{\ln(100/105) + \left(\frac{r}{365} + 0.02^2/2\right)20}{0.02\sqrt{20}} = -0.49464
\]

\[
d_2 = d_1 - 0.02\sqrt{20} = -0.58409
\]

\[
C = 100N(-0.04424) - 105e^{-0.01(20/365)}N(-0.58409) = 1.70
\]

Using an Option Value Calculator to Calculate this same Value

<table>
<thead>
<tr>
<th>Call Option Price Calculator (Daily Volatility)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock symbol: Trial</td>
</tr>
<tr>
<td>Call option: May</td>
</tr>
<tr>
<td>Date Today: 4/26/2011</td>
</tr>
<tr>
<td>Expiration Date: 5/16/2011</td>
</tr>
<tr>
<td>DTM: 20</td>
</tr>
<tr>
<td>Stock Price: 100.00</td>
</tr>
<tr>
<td>Strike Price: 105.00</td>
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<tr>
<td>Daily Volatility: 0.02</td>
</tr>
<tr>
<td>Interest Rate: 0.01</td>
</tr>
<tr>
<td>Time: 20</td>
</tr>
<tr>
<td>d1 Numerator: -0.04424</td>
</tr>
<tr>
<td>Duration Volatility: 0.08944</td>
</tr>
<tr>
<td>Delta N(d1): 0.3104</td>
</tr>
<tr>
<td>N(d2): 0.2798</td>
</tr>
<tr>
<td>Option Price: 1.70</td>
</tr>
<tr>
<td>Option Premium: 1.70</td>
</tr>
</tbody>
</table>

\[
=LN(SP/KP)+(IR+(DV*DV)/2)*DTM/365
\]

\[
=LN(SP/KP)+((IR/365)+(DV*DV)/2)*DTM
\]

\[
=LN(100/105)+\left(\frac{r}{365} + 0.02^2/2\right)20
\]

\[
=\ln(100/105) + \left(\frac{0.01}{365} + 0.02^2/2\right)20
\]

\[
=LN(SP/KP)+((IR/365)+(DV*DV)/2)*DTM
\]

\[
=NORMSDIST(NUM/DUV)
\]