Compounding, Discounting, and unbiased Growth Rates

Near Deb’s school in Southern Colorado. An example of slow growth.

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Why we use compounding and discounting approaches

When we work with valuation methods or the pricing of options, futures, or forward contracts, we are in an environment where payment is separated by time from delivery.

When we work with yield-bearing financial assets like bonds (not directly a subject in this class) we are trying to find a value for a cashflow stream that projects into the future.

Compounding and (especially) discounting formulas provide us with the tools necessary to place a current value on these contracts or assets.
Discounting summation formulas

These are used when there are multiple payments, typically of the same value, over many time periods.

Example: The California lottery pays the winner 20 equal annual installments (1/20 of the announced prize), the first immediately and the last in 19 years. What would be the present value of a $40 million prize (which implies a $2 million cash payment annually) when the risk free interest rate is 6%?

\[ PV_0 = \sum_{i=0}^{19} 2(1.06)^{-i} \]

This may also be written

\[ PV_0 = 2 + 2 \left( \frac{1}{1.06} \right) + 2 \left( \frac{1}{1.06} \right)^2 + \ldots + 2 \left( \frac{1}{1.06} \right)^{19} \]

Status check

- What we learned in E104 (or elsewhere) ..
  - In situations where we expect a discrete (countable) number of payments in the future, we have seen the formulas for estimating their present value.

- Where we go next ..
  - We now must consider the continuous compounding case. Note: if you have money in an account that is “compounded daily” this is effectively an example of continuous compounding.
Continuous compounding

If we are thinking about variables that change over time, like the market value of stock or the value of a home, we can say that the future value of the variable (typically an asset) is equal to the original value of the variable times an instantaneous growth rate expressed by the formula:

\[ X_t = X_0 e^{rt} \]

where
- \( e = 2.718281828 \)
- \( r = \) continuous growth rate
- \( t = \) time (in any stipulated interval, like days or years)
- \( X_t = \) value of \( X \) at time \( t \)
- \( X_0 = \) initial value of \( X \)

Example

What is the value of a house if it was purchased at $120,000 and has been growing in value at the rate of 8% annually compounded daily for a period of 4 years and 3 months?

\[ X_t = 168.59 = 120e^{0.08 \times 4.25} \]

Note: Although compounding daily is discrete (there are 365 payments per year) the continuous compounding formula is accurate up to two decimal points.
Continuous discounting

What is the present value of an asset or a contract that will be delivered in the future if we use the continuous compounding interest rate of $r$?

\[ X_0 = X_t e^{-rt} \]

where
- $e, r,$ and $t$ are the same
- $X_0 =$ present value now
- $X_t =$ delivery value in the future

Note: This formula and the continuous compounding formula are used extensively in the valuation of futures and forward contracts, as well as generally in financial valuations.

Example

What is the present value of $100,000 to be delivered in four years and six months if the present short-term money rate compounded daily is 3.45%?

\[ X_0 = \text{answer} = 100e^{-0.0345 \cdot 4.5} \]

$85,620
Looking ahead .. using natural logs

Remembering the relationship between natural logs and the “magic number” $e = (2.718281828)$:

\[
\text{if } X = e^n \text{ then } \ln X = n
\]

and

\[
\text{if } X = Ae^n \text{ then } \ln X = \ln A + n
\]

This means that

\[
\text{if } X_t = X_0e^{rt} \text{ then } \ln X_t = B + rt
\]

and

\[
\frac{\partial}{\partial t} (\ln X_t) = r
\]

Data conversion: continuous growth rates

1. $X_f = X_p e^{rt}$

   $X_f$ may be a spot price, $X_p$ may be a future spot price

2. $\ln X_f = \ln X_p + rt$

   $r$ is typically the annualized growth rate unless otherwise stipulated.

3. $(\ln X_f - \ln X_p)(1/t) = r$

4. $\ln \left( \frac{X_f}{X_p} \right) \times \left( \frac{1}{t} \right) = r$

5. also calculated as $\ln X_t - \ln X_{t-1}$

If these are weekly data, this value is $1/(1/52) = 52$.

If these data are for 3 years, this value is $(1/3) = 0.333$. 

Mudd Finance
... for example

Consider natural gas, $ per MMBTU:

\[ X_{march} = 2.756 \text{ and } X_{july} = 3.058 \text{ then} \]
\[ \ln(3.058/2.756) \times 12/4 = 31.19\% \text{ annualized}, \]
\[ [\ln(3.058) - \ln(2.756)] \times 12/4 = 31.19\% \]

Conversion rates

From continuous to discrete:

\[ r_d = m(e^{r_c/m} - 1) \]

From discrete to continuous:

\[ r_c = m \ln\left(1 + \frac{r_d}{m}\right) \]

From discrete to discrete, different time periods:

\[ r_{m_2} = \left[\left(1 + \frac{r_{m_1}}{m_1}\right)^{m_1/m_2} - 1\right]m_2 \]

\[ m = \text{number of periods in a year; i.e. for quarterly } m = 4. \]
Example ...

1. Convert a 6% interest rate compounded annually to a continuous rate.
2. Convert a 6% interest rate compounded quarterly to a continuous rate.

\[ r_c = \ln(1.06) = 5.83\% \]
\[ r_c = 4\ln\left(1 + \frac{0.06}{4}\right) = 5.955\% \]

Reference slide.

Three alternative growth rates
(point by point estimation from time-series data)

<table>
<thead>
<tr>
<th>t</th>
<th>x</th>
<th>Discrete: ( \frac{X_{t+1} - X_t}{X_{t+1}} = \frac{X_{t+1}}{X_t} - 1 )</th>
<th>Geometric Discrete: ( \frac{X_{t+1}}{X_t} )</th>
<th>Continuous (log): ( \ln\left(\frac{X_{t+1}}{X_t}\right) = \ln(X_{t+1}) - \ln(X_t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>t = 0</td>
<td>( x_0 = 100 )</td>
<td>0.15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>t = 1</td>
<td>( x_1 = 115 )</td>
<td>1.15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>etc.</td>
<td></td>
<td></td>
<td>0.13976</td>
<td></td>
</tr>
</tbody>
</table>

Working with raw time-series data, we have to transform it to growth rates, but which?
... calculating geometric mean and geometric standard deviation (from discrete-converted data)

Geometric Mean value of growth rate:

\[ \mu_{gx} = \left( \prod_{i=1}^{n} x_i \right)^{1/n} \]

or

\[ \mu_{gx} = e^{\frac{1}{n} \sum_{i=1}^{n} \ln(x_i)} \]

Geometric standard deviation of the growth rate:

\[ \sigma_{gx} = e^{\left( \frac{\sum_{i=1}^{n} \left[ \ln(x_i) - \ln(\mu_{gx}) \right]^2}{t} \right)} \]

Remember – the growth rate will be expressed as 1.15 rather than 0.15.

... calculating mean, standard deviation and variance from the log-converted data

Mean value of growth rate:

\[ \overline{R}_p = \frac{\sum_{i=1}^{n} R_i}{n} = \mu_p \]

Variance of the growth rate:

\[ V_p = \frac{\sum_{i=1}^{n} (R_i - \overline{R})^2}{n} = \sigma_p^2 \]

Standard deviation of growth rate:

\[ SD_p = \sqrt{V_p} = \sigma_p \]

Note: these estimators assume that you are using a Gaussian (normal) distribution. Equally important – these classical estimators cannot be used on data transformed into discrete growth rates!

Note: A good source for reviewing statistical applications is the National Institute of Standards and Technology Engineering Statistics Handbook, available online at http://www.itl.nist.gov/div898/handbook/
A problem that we need to think about ...

When you transform raw time-series data to discrete or geometric discrete growth rates, both the traditional arithmetic mean and standard deviation are biased downward and therefore unusable. Why? You have converted the data into a geometric series and because of that this necessary condition for the absence of bias is violated:

For observations \( X_0 \) (the first) to \( X_f \) (the final) where \( \alpha_i \) is the discrete growth rate for time \( i \) and \( \mu \) is the arithmetic mean of these numbers (the alphas):

\[
X_f = X_0 (1 + \mu)^t \neq X_0 \prod_{i=1}^{t} (1 + \alpha_i)
\]

However, if we take the arithmetic mean and standard deviation of the continuous growth rate, the estimators are not biased because the following condition is met (where \( \alpha \) is now the continuous growth rate):

Remember:
\[
\begin{align*}
X_1 &= X_0 (1 + \alpha_1) \\
X_2 &= X_1 (1 + \alpha_2) = X_0 (1 + \alpha_1)(1 + \alpha_2) \\
X_3 &= X_2 (1 + \alpha_3) = X_0 (1 + \alpha_1)(1 + \alpha_2)(1 + \alpha_3)
\end{align*}
\]

Why we can’t use the arithmetic mean:

Suppose you had an $100 investment that earned, over five years, these yields:

<table>
<thead>
<tr>
<th>Year</th>
<th>Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15%</td>
</tr>
<tr>
<td>2</td>
<td>20%</td>
</tr>
<tr>
<td>3</td>
<td>30%</td>
</tr>
<tr>
<td>4</td>
<td>-20%</td>
</tr>
<tr>
<td>5</td>
<td>25%</td>
</tr>
</tbody>
</table>

The arithmetic mean of these returns is 14%.

So will your investment at the end of five years be worth

\[
100 \times 1.14^5 = \$192.54?
\]

No. Using the numbers directly, your investment will be worth:

\[
100 \times (1.15) \times (1.20) \times (1.30) \times (0.80) \times (1.25) = \$179.40
\]

You can’t take the mean of the historical value of discrete growth rates to calculate the actual growth from beginning to end (or estimate its continuation). It will be biased!
One more example:

Suppose a stock index went from 100 to 50, which we would describe as a percentage fall of 50%, then from 50 to 75, which we would describe as a percentage increase of 50%. So because it went down by 50% then up by 50%, are we back where we started? Hardly!

Note that:

\[ 100 \times (0.866)^2 = 75 \]

and

\[ \ln \left( \frac{75}{100} \right)^{\frac{1}{2}} = -0.1438 \]

The convenient additive and mean-calculcation feature of log growth rates

1. \[ X_1 = X_0 e^{r_1} \]
2. \[ X_2 = X_1 e^{r_2} = X_0 e^{r_1} e^{r_2} = X_0 e^{(r_1 + r_2)} \] etc.

generally ...

3. \[ X_t = X_0 e^{\frac{\sum_{i=1}^{t} r_i}{t}} = X_0 e^{t(Mean \ r)} \]

which means that we can take mean continuous growth rates and compound and discount, including compound forward for growth estimates, with bias.
First steps in estimating measures of daily continuous growth rates

Given some \textit{sample} from a \textit{population} set of prices, such as the daily closing price of CSCO, for us the first step is to convert the data to continuous growth rates:

\[ CGR_{px} = R_p = \ln \left( \frac{P_t}{P_{t-1}} \right) \]

for each paired observation.

In Excel:

<table>
<thead>
<tr>
<th>Sample</th>
<th>Price</th>
<th>CGRP</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>22.34</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>22.86</td>
<td>0.0230</td>
</tr>
<tr>
<td>3</td>
<td>23.01</td>
<td>0.0065</td>
</tr>
<tr>
<td>4</td>
<td>22.79</td>
<td>-0.0096</td>
</tr>
<tr>
<td>5</td>
<td>23.41</td>
<td>0.0268</td>
</tr>
<tr>
<td>6</td>
<td>23.56</td>
<td>0.0064</td>
</tr>
</tbody>
</table>

Original sample (in part) \hspace{1cm} Converted data

Period Conversion
(of growth rates)

In this class we are usually using daily data and are content to estimate and use daily continuous growth rates, which requires no further conversion. If you are using weekly data then you are calculating weekly continuous growth rates. Also, much of the literature and Hull’s book consistently refers to annual continuous growth rates (an approach I don’t use of like).

Use the following convention for period conversion. \textbf{Important note:} If the data shows only 5 observations per week or 252 observations per year, then use those numbers rather than 7 or 365.

\[ Period \ GR = [Mean \ Daily \ GR \times Days \ in \ Period] \]

\[ Annual \ GR = 252 \times [M]DGR \]
Head-start on the large HW1 set

As soon as you learn about risk proxies, our next segment, you will be given a very large HW set in Excel. In that you will be asked to download one year of daily data in SPY, then convert it to daily continuous growth rates, which you should now be able to do.

You need to know how to download historical data from finance.yahoo.

Watch this video to see how to do that (and to convert it to the proper format): http://www.screencast.com/t/0l3BuNoT

Topical ... so far, sluggish market in 2014

Earnings were stellar in 2013 and margins were fat, and the largest corporations are cash-rich ($7t)!

Growth is picking up very slightly but the outlook is still uncertain, and the FRS is supposed to start tapering QE3.

Some of the emerging nations, in particular Brazil, Argentina, India, Turkey, plus Australia and maybe Canada, are suffering from capital flow and exchange rate problems. El-Erian quit Pimco?? We return to these later.