Extending Castigliano's Theorems to Model the Behavior of Coupled Systems

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Extensions of the Castigliano theorems are developed in the context of modeling the behavior of both discrete coupled linear systems and various coupled beams. It is shown that the minimization of the displacement of a parallel system determined from Castigliano's second theorem can be used to formally define the apportionment of loads among the system elements, while the minimization of the load in a series system determined from Castigliano's first theorem can be used to formally define the apportionment of the displacement among the system elements. These extensions provide a means for apportioning loads in coupled continuous systems, as will be shown for the cases of coupled cantilever Timoshenko beams supporting discrete and continuous loads.
[DOI: 10.1115/1.4001716]

1 Introduction

The work presented herein has its roots in the development of models of buildings as collections of beam components that can be used in preliminary design and for estimating dynamic behavior (see, for example, Refs. [1–6]). Static models of analysis often assume series models of behavior, wherein bending and shear deflections of beam deflection (or story drifts) add to form the total deflection or drift [4]. However, while the shear and bending deflections of individual beam elements add in *series* (as in a Timoshenko beam), the aggregation of the beam components invariably requires a *parallel* formulation because that aggregation typically requires that the building deflection or drift is the same for each of its component elements, whether those elements are individually in (pure) shear or bending, or a series combination of the two. For example, the dynamic response of a building comprised of a shear core and a bent tube is modeled in a parallel combination of a Euler beam that is constrained to deflect equally with a companion shear beam (e.g., Refs. [6–8]). In such models, since the component deflections are constrained to be equal to each other (and thus, to the total deflection), the bending and shear stiffnesses add linearly, and the external load is distributed among the individual components in proportion to each component's stiffness as a fraction of the total stiffness.

Now, while the apportionment of an external load among the parallel components is easily done for discrete models, wherein the external load is a single "concentrated" load, it is not obvious how that distribution can be implemented for a load that is spatially distributed over the beam's length (or a building model's height). This situation is analogous to the power of the discretization inherent in Castigliano's second theorem, in which the deflection under a concentrated load is easily calculated, while the deflection under a distributed load requires the additional introduction of a phantom force [9]. In what follows below, an extension of Castigliano's second theorem is proposed to calculate load apportionment and consequent point deflections for continuous models. Further, the individual or component beams that are used to construct building such models include elementary Euler–Bernoulli beams in bending, shear beams, and Timoshenko beams, in which both bending and shear deflections are added in series.

Contributed by the Applied Mechanics of ASME for publication in the Journal of Applied Mechanics. Manuscript received September 29, 2009; final manuscript received March 11, 2010; accepted manuscript posted May 5, 2010; published online August 17, 2010. Assoc. Editor: Martin Ostoja-Starzewski.

While the motivation described stems from the modeling of high rise buildings in terms of beams, the proposed methodology is applicable to any parallel concatenation of elements. Moreover, a corresponding extension of Castigliano's first theorem that permits the calculation of the apportionment of displacements when elements are arrayed in series is also shown. The extensions for both the Castigliano theorems are developed immediately below for discrete systems, after which, the tip deflection for a parallel pairing of two Timoshenko cantilevers is found when they are subjected to, respectively, a uniform load q_0, a tip load P, and tip moment M^*.

2 Extensions of the Castigliano Theorems for Discrete Systems

Consider an elementary discrete system of two linear springs ostensibly connected in parallel. If each spring has stiffness k_i and supports an internal spring force F_i, the complementary energy contained in the system is

$$U^* = \sum_{i=1}^{2} \frac{F_i^2}{2k_i} \tag{1}$$

Note that the analysis that follows is readily extended to systems with n springs. If an external load P is applied to the system, the respective internal spring forces are assumed to be

$$F_1 \equiv rP \quad \text{and} \quad F_2 = (1-r)P \tag{2}$$

where r is a positive constant, such that $0 \le r \le 1$. Then the system's complementary energy (1) becomes

$$U^* = \frac{1}{2}\left(\frac{r^2}{k_1} + \frac{(1-r)^2}{k_2}\right)P^2 \tag{3}$$

Castigliano's second theorem states that the resulting system displacement is found as

$$\delta_{\text{par}} = \frac{dU^*}{dP} = \left(\frac{r^2}{k_1} + \frac{(1-r)^2}{k_2}\right)P \tag{4}$$

Equation (4) represents the statement of Castigliano's second theorem [10], which can be derived by minimizing the total complementary energy Π^* [9]. Thus, the force that minimizes the total complementary energy produces a compatible displacement as given by Eq. (4).

But how is the value of r to be determined? It is easily shown that the apportionment factor r is that which minimizes the system displacement δ_{par}. Thus, the value of r for which

$$\frac{d\delta_{\text{par}}}{dr} = 2\left(\frac{r}{k_1} - \frac{(1-r)}{k_2}\right)P = 0 \tag{5}$$

is

$$r = \frac{k_1}{k_1 + k_2} \tag{6}$$

from which, it then follows that

$$\delta_{\text{par}} = \frac{P}{k_1 + k_2} \tag{7}$$

Equation (7) is a familiar and expected result, although its derivation is not. Further, it is evident that the value of r in Eq. (5) is actually a minimum since

$$\frac{d^2\delta_{\text{par}}}{dr^2} = 2\left(\frac{1}{k_1} + \frac{1}{k_2}\right)P > 0 \tag{8}$$

Thus, the following extension of Castigliano's second theorem emerges:

> The minimum of the (compatible) displacement of a parallel system as determined by Castigliano's second theorem corresponds to an equilibrium apportioning of the loads carried by the elements in that system.

A physical interpretation of this result is as follows. Note that the system displacement (7) found by this process means that each spring has the same displacement, that is

$$\delta_1 = \left(\frac{F_1 = rP}{k_1}\right) \equiv \delta_{\text{par}} \quad \text{and} \quad \delta_2 = \frac{(F_2 = (1-r)P)}{k_2} \equiv \delta_{\text{par}} \tag{9}$$

The value of r thus determined also guarantees that the displacement of each element in this parallel system will be the same. Thus, it ought to be no surprise that the apportionment factor thus determined produces a familiar, seemingly self-evident result.

This extension of Castigliano's second theorem is readily extended to systems with n degrees of freedom, in which case, the corresponding complementary energy is written in terms of n distribution factors for which $\sum_{i=1}^{n} r_i = 1$

$$U_n^* = \sum_{i=1}^{n} \frac{F_i^2}{2k_i} = \frac{1}{2}\left(\frac{r_1^2}{k_1} + \frac{r_2^2}{k_2} + \cdots + \frac{(1 - r_1 - r_2 \cdots - r_{n-1})^2}{k_n}\right)P^2 \tag{10a}$$

which can also be written in the form

$$U_n^* = \left(\frac{k_n}{k_1}r_1^2 + \frac{k_n}{k_2}r_2^2 + \cdots + (r_{n-1} + r_{n-2} + \cdots + r_1 - 1)^2\right)\frac{P^2}{2k_n} \tag{10b}$$

Once again, Castigliano's second theorem produces the system displacement, in this case

$$\delta_{\text{par}/n} = \frac{dU_n^*}{dP} = \left(\frac{k_n}{k_1}r_1^2 + \frac{k_n}{k_2}r_2^2 + \cdots + (r_n + r_{n-1} + \cdots + r_1 - 1)^2\right)\frac{P}{k_n} \tag{11}$$

The minimization of the system displacement $\delta_{\text{par}/n}$ with respect to the factors $r_1, r_2, \cdots, r_{n-1}$ produces the following system of n equations for those factors:

$$\begin{bmatrix} (k_n/k_1+1) & 1 & \cdots & 1 & 1 \\ 1 & (k_n/k_2+1) & \cdots & 1 & 1 \\ 1 & 1 & \ddots & 1 & 1 \\ 1 & 1 & \cdots & (k_n/k_{n-2}+1) & 1 \\ 1 & 1 & \cdots & 1 & (k_n/k_{n-1}+1) \end{bmatrix} \times \begin{Bmatrix} r_1 \\ r_2 \\ \vdots \\ r_{n-2} \\ r_{n-1} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{Bmatrix} \tag{12}$$

The simplicity of Eq. (12) enables it to be seen by inspection that the (unique) solution to this linear system is (a formal derivation may be found in the Appendix)

$$\begin{Bmatrix} r_1 \\ r_2 \\ \vdots \\ r_{n-2} \\ r_{n-1} \end{Bmatrix} = \left(\frac{1}{\sum_{i=1}^{n} k_i}\right) \begin{Bmatrix} k_1 \\ k_2 \\ \vdots \\ k_{n-2} \\ k_{n-1} \end{Bmatrix} \tag{13}$$

because the substitution of Eq. (13) into Eq. (12) yields

$$\begin{bmatrix} (k_n/k_1+1) & 1 & \cdots & 1 & 1 \\ 1 & (k_n/k_2+1) & \cdots & 1 & 1 \\ 1 & 1 & \ddots & 1 & 1 \\ 1 & 1 & \cdots & (k_n/k_{n-2}+1) & 1 \\ 1 & 1 & \cdots & 1 & (k_n/k_{n-1}+1) \end{bmatrix} \times \left(\frac{1}{\sum_{i=1}^{n} k_i}\right) \begin{Bmatrix} k_1 \\ k_2 \\ \vdots \\ k_{n-2} \\ k_{n-1} \end{Bmatrix} = \left(\frac{1}{\sum_{i=1}^{n} k_i}\right) \times \begin{Bmatrix} (k_n + k_1) + k_2 + \cdots + k_{n-2} + k_{n-1} \\ k_1 + (k_n + k_2) + \cdots + k_{n-2} + k_{n-1} \\ k_1 + k_2 + \cdots + k_n + \cdots + k_{n-2} + k_{n-1} \\ k_1 + k_2 + \cdots + (k_n + k_{n-2}) + k_{n-1} \\ k_1 + k_2 + \cdots + k_{n-2} + (k_n + k_{n-1}) \end{Bmatrix} \equiv \begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{Bmatrix} \tag{14}$$

Equation (13) clearly shows that the load apportionment factors for each of the n elements in the system are, in scalar form

$$r_j = \frac{k_j}{\sum_{i=1}^{n} k_i}, \quad j = 1, 2, \cdots, n-1 \tag{15}$$

Finally, the Hessian matrix of the quadratic form (11) that makes up the displacement $\delta_{\text{par}/n}$ is clearly a positive constant, that is

$$\begin{bmatrix} (k_n/k_1+1) & 1 & \cdots & 1 & 1 \\ 1 & (k_n/k_2+1) & \cdots & 1 & 1 \\ 1 & 1 & \ddots & 1 & 1 \\ 1 & 1 & \cdots & (k_n/k_{n-2}+1) & 1 \\ 1 & 1 & \cdots & 1 & (k_n/k_{n-1}+1) \end{bmatrix} > 0 \tag{16}$$

Hence, the displacements of each element are the same minimum displacement

$$\delta_i = \left(\frac{F_i = r_i P}{k_i} \right) \equiv \delta_{\text{par}/n} \tag{17}$$

It is worth noting again that in contrast with traditional descriptions in which a common displacement for parallel systems is imposed ab initio as a constraint, this extension of Castigliano's second theorem has that common displacement emerging as an end result.

A corresponding extension of Castigliano's first theorem can be developed for the displacements that result when a common load is applied to a system of springs in series. Thus, if each spring has stiffness k_i and extends an amount δ_i, the strain energy stored in the system is

$$U = \sum_{i=1}^{2} \frac{1}{2} k_i \delta_i^2 \tag{18}$$

If an external load P is applied to the system, the respective spring displacements can be assumed to be

$$\delta_1 \equiv r\delta \quad \text{and} \quad \delta_2 = (1-r)\delta \tag{19}$$

where r is again a positive constant, such that $0 \leq r \leq 1$. Then the system's stored energy (1) becomes

$$U = \frac{1}{2}(k_1 r^2 + k_2(1-r)^2)\delta^2 \tag{20}$$

Castigliano's first theorem states that the resulting force supported by the system is

$$P_{\text{ser}} = \frac{dU}{d\delta} = (k_1 r^2 + k_2(1-r)^2)\delta \tag{21}$$

As with the discussion above (following Eq. (4)), it is well known that Eq. (21) represents the statement of Castigliano's first theorem [10], which can be derived by minimizing the total potential energy Π [9]. Thus, the displacement that minimizes the total potential energy produces the equilibrium statement, as given by Eq. (21).

Also similarly, the value of the apportionment factor r can be determined as that, which minimizes the system load P_{ser}, that is, the one for which

$$\frac{dP_{\text{ser}}}{dr} = 2(k_1 r - k_2(1-r))\delta = 0 \tag{22}$$

which results in

$$r = \frac{k_2}{k_1 + k_2} \tag{23}$$

and from which it follows that

$$P_{\text{ser}} = \left(\frac{1}{k_1} + \frac{1}{k_2} \right)^{-1} \delta \tag{24}$$

Again, Eq. (24) is familiar and expected, and here too, it is evident that the value of r in Eq. (23) actually corresponds to a minimum of P_{ser} since

$$\frac{d^2 P_{\text{ser}}}{dr^2} = 2(k_1 + k_2)\delta > 0 \tag{25}$$

Thus, an extension of Castigliano's first theorem now emerges:

> The minimum of the (equilibrium) force in a series system as determined by Castigliano's first theorem corresponds to a compatible apportioning of the displacements of the elements in that system.

A physical interpretation of this result is also possible. Note that the system load (24) found by this process means that each spring supports or transmits the same load, that is

$$F_1 = k_1(\delta_1 = r\delta) = P_{\text{ser}} \quad \text{and} \quad F_2 = k_2(\delta_2 = (1-r)\delta) = P_{\text{ser}} \tag{26}$$

The value of r thus determined also guarantees that the force carried by each element in this parallel system will be the same. Thus, it ought to be no surprise that the apportionment factor thus determined produces still another familiar, seemingly self-evident result.

3 Castigliano's Second Theorem for Coupled Timoshenko Beams

The foregoing extension of Castigliano's second theorem is now used to examine the apportioning of loads in a continuous coupled system, namely, that of coupled Timoshenko beams. The shear and moment resultants of beams under a distributed load $q(x)$ are governed by classical equations of equilibrium [10]

$$\frac{dV(x)}{dx} + q(x) = 0$$

$$\frac{dM(x)}{dx} - V(x) = 0 \tag{27}$$

Which, for statically determinate beams, are straightforwardly integrated as (with $x=0$ taken at the cantilever root) to yield

$$V(x) = -\int_H^x q(\xi)d\xi$$

$$M(x) = \int_H^x V(\eta)d\eta = -\int_H^x \int_H^\eta q(\xi)d\xi d\eta \tag{28}$$

Note that Eqs. (27) and (28) are valid for elementary Euler beams in bending, shear beams, and Timoshenko beams.

Consider now the case of a pair of Timoshenko beams supporting a load $q(x)$ distributed along its length. The complementary energy for such a coupled cantilever system is a straightforward extension of the complementary energy for a single Timoshenko beam that incorporates the energy due to both bending and shear in each beam [10]

$$U_{2\text{Timo}}^* = \sum_{i=1}^{2} [U_{si}^* + U_{bi}^*] = \sum_{i=1}^{2} \left[\int_0^L \frac{(V_i(x))^2}{2S_i} dx + \int_0^L \frac{(M_i(x))^2}{2B_i} dx \right] \tag{29}$$

Each of the beams has its own bending stiffness $B_i = E_i I_i$ and shear stiffness $S_i = k_i G_i A_i$. Let the distributed load applied to the cantilever system be apportioned between the two beams, such that $q(x) = q_1(x) + q_2(x)$, and include also a (similarly apportioned) tip load P that will be used as a phantom load for determining the cantilever tip's deflection. The apportioned applied and (phantom) tip loads are then

$$q_i(x) \equiv r_i q(x) \quad \text{and} \quad P_i \equiv r_i P \tag{30}$$

In view of Eq. (30), the moment and shear resultants in each Timoshenko beam can be written as

$$V_i(x) = r_i \left[-\int_0^x q(\xi)d\xi + P \right] \equiv r_i(V(x) + P)$$

$$M_i(x) = r_i \left[-\int_0^x \int_0^\eta q(\xi)d\xi d\eta + P(x-H) \right]$$

$$\equiv r_i(M(x) + P(x-H)) \tag{31}$$

Now, the tip deflection for the distributed load (only) then follows from Castigliano's second theorem [9] as

$$w_{q(x)}(L) = \left. \frac{dU^*_{2\mathrm{Timo}}}{dP} \right|_{P\to 0} = \sum_{i=1}^{2} \left[\int_0^L \frac{V_i(x)}{S_i}\frac{dV_i(x)}{dP}dx + \int_0^L \frac{M_i(x)}{B_i}\frac{dM_i(x)}{dP}dx \right] \quad (32)$$

so that the substitution of Eq. (31) into Eq. (32) produces the tip deflection in the form

$$w_{q(x)}(L) = \sum_{i=1}^{2} \left[\left(\frac{r_i^2}{S_i}\right)\left(\int_0^L V(x)dx + \left(\frac{\lambda_i^2}{L^2}\right)\int_0^L M(x)(x-L)dx \right) \right] \quad (33)$$

where a dimensionless factor has been introduced in Eq. (33)

$$\lambda_i^2 \equiv \frac{L^2 S_i}{B_i} = \frac{k_i G_i}{E_i}\left(\frac{A_i L^2}{I_i}\right) \quad (34)$$

Note that the ratio (34) is defined for each beam individually, and thus, are not to be confused with a similar dimensionless parameter α defined in Refs. [7,8] to couple a shear beam to a Euler beam to model the behavior of various combinations of frames and shear walls.

Note further that Eq. (33) contains two constants that represent weighted averages of the shear and moment resultants, that is

$$\bar{V} \triangleq \int_0^L V(x)dx \quad \text{and} \quad \bar{M} \triangleq \int_0^L M(x)(x-L)dx \quad (35)$$

so that Eq. (33) can also be written as

$$w_{q(x)}(L) = \bar{V}\sum_{i=1}^{2} \left[\left(\frac{r_i^2}{S_i}\right)\left(1 + \lambda_i^2\left(\frac{\bar{M}}{L^2\bar{V}}\right)\right) \right] \quad (36)$$

Equation (36) can be cast in terms of an effective stiffness of each Timoshenko beam that corresponds to the particular loading that produces the weighted stress resultants $\bar{V}$ and $\bar{M}$

$$K_{Ti} = \frac{S_i}{1 + \lambda_i^2\left(\dfrac{\bar{M}}{L^2\bar{V}}\right)} \equiv \frac{S_i}{1 + \lambda_i^2/a^2} \quad (37)$$

where a^2 is a constant fraction that reflects those weighted stress resultants

$$a^2 \triangleq \frac{L^2\bar{V}}{\bar{M}} \quad (38)$$

Then, in view of Eqs. (36) and (37), the tip deflection assumes its nearly final form

$$w_{q(x)}(L) = \bar{V}\sum_{i=1}^{2}\left(\frac{r_i^2}{K_{Ti}}\right) \quad (39)$$

Equation (39) represents the discretized tip deflection of a cantilever system of (in this case) two Timoshenko beams subjected to the load that produces the weighted shear $\bar{V}$. How is that load to be distributed among the parallel elements that make up the system?

For the case of (only) two coupled beams, there is only one independent apportionment factor, that is

$$r_1 = r \quad \text{and} \quad r_2 = (1-r) \quad (40)$$

so that Eq. (39) then becomes

$$w_{q(x)}(L) = \left(\frac{r^2\bar{V}}{K_{T1}}\right) + \left(\frac{(1-r)^2\bar{V}}{K_{T2}}\right) \quad (41)$$

Following the procedure proposed above for discrete systems, Eq. (41) can be minimized to determine the value of the apportionment factor r that minimizes the tip deflection of the cantilever system. This, in turn, produces a compatible tip deflection of the system that properly preserves equilibrium of the applied loading with the forces carried in the two Timoshenko beams. In that instance, the minimizing value of the apportionment factor is easily found to be

$$r_{\min} = \frac{K_{T1}}{K_{T1}+K_{T2}} = \frac{1}{1+\dfrac{K_{T2}}{K_{T1}}} = \frac{1}{1+\dfrac{1+\lambda_1^2/a^2}{1+\lambda_2^2/a^2}\left(\dfrac{S_2}{S_1}\right)} \quad (42)$$

and the corresponding tip deflection is

$$w_{q(x)}(L) = \frac{\bar{V}}{K_{T1}+K_{T2}} \quad (43)$$

Of course, Eq. (43) is a very familiar form, albeit derived with the formal apportionment approach derived from Castigliano's second theorem and presented in a general form that is applicable to any load $q(x)$ applied along the length of the cantilever system. Equation (43) can also be written to reflect the four (for the case of two Timoshenko beams) independent stiffness terms (i.e., S_1,S_2,B_1,B_2 or $S_1,S_2,\lambda_1^2,\lambda_2^2$) in two equivalent, symmetric forms

$$w_{q(x)}(L) = \frac{1+\lambda_1^2/a^2}{1+\dfrac{1+\lambda_1^2/a^2}{1+\lambda_2^2/a^2}\left(\dfrac{S_2}{S_1}\right)}\left(\frac{\bar{V}}{S_1}\right) = \frac{1+\lambda_2^2/a^2}{1+\dfrac{1+\lambda_2^2/a^2}{1+\lambda_1^2/a^2}\left(\dfrac{S_1}{S_2}\right)}\left(\frac{\bar{V}}{S_2}\right) \quad (44)$$

Equations (44) thus represent the tip deflection of a coupled pair of cantilevered Timoshenko beams as a (symmetric) function of the two dimensionless ratios λ_i^2. Note that, as mentioned earlier, these ratios refer to the properties of each of the Timoshenko beams independently.

Special cases are readily derived. Consider, for example, a Euler beam $(i=1)$ in bending coupled with a simple shear beam $(i=2)$. A Euler beam requires that $S_1\to\infty$ and $\lambda_1^2\to\infty$, while $\lambda_1^2/S_1\to L^2/B_1$, and a shear beam requires $B_2\to\infty$ and $\lambda_2^2\to 0$, while $\lambda_2^2 B_2\to L^2 S_2$. Then the apportionment ratio for the Euler-shear combination is

$$r_{\min} = \frac{a^2 B_1/L^3}{a^2 B_1/L^3 + S_1/L} \quad (45)$$

while the concomitant tip deflection is

$$w_{q(x)}(L) = \frac{\bar{V}/L}{a^2 B_1/L^3 + S_1/L} \quad (46)$$

Equations (45) and (46) clearly reflect the parallel stiffnesses of the Euler $(a^2 B_1/L^3)$ and shear (S_1/L) beams.

To illustrate how the tip deflection varies with the actual load distribution, consider the case of a uniform distributed load, that is, $q(x)=q_0$, a constant. The shear and moment resultants follow from Eq. (20) as

$$V(x) = -q_0(x-L)$$

$$M(x) = -\frac{q_0(x-L)^2}{2} \quad (47)$$

Then the parameters defined in Eqs. (35) and (38) are found to be

$$\bar{V} = \frac{q_0 L^2}{2}, \quad \bar{M} = \frac{q_0 L^4}{8}, \quad a^2 = 4 \tag{48}$$

so that the tip deflection (or building or story drift) under a uniform load is

$$w_{q_0}(L) = \frac{1 + \lambda_1^2/4}{1 + \frac{1 + \lambda_1^2/4}{1 + \lambda_2^2/4}\left(\frac{S_2}{S_1}\right)}\left(\frac{q_0 L^2}{2S_1}\right) = \frac{1 + \lambda_2^2/4}{1 + \frac{1 + \lambda_2^2/4}{1 + \lambda_1^2/4}\left(\frac{S_1}{S_2}\right)}\left(\frac{q_0 L^2}{2S_2}\right) \tag{49}$$

By way of comparison, consider a tip load P carried by two Timoshenko beams, in which case, the moment and shear resultants are found from Eq. (20)

$$M(x) = P(x - L)$$

$$V(x) = P \tag{50}$$

Then the parameters defined in Eqs. (35) and (38) are found to be

$$\bar{V} = PL, \quad \bar{M} = \frac{PL^3}{3}, \quad a^2 = 3 \tag{51}$$

so that the tip deflection (or building or story drift) under a tip load is

$$w_{q_0}(L) = \frac{1 + \lambda_1^2/3}{1 + \frac{1 + \lambda_1^2/3}{1 + \lambda_2^2/3}\left(\frac{S_2}{S_1}\right)}\left(\frac{PL}{S_1}\right) = \frac{1 + \lambda_2^2/3}{1 + \frac{1 + \lambda_2^2/3}{1 + \lambda_1^2/3}\left(\frac{S_1}{S_2}\right)}\left(\frac{PL}{S_2}\right) \tag{52}$$

Finally, consider a coupled pair of Timoshenko beams loaded by a tip moment M^*. In this case, the shear force $V(x)=0$, so that Eq. (33) must be modified in the light of the definition (34) to read

$$w_{q(x)}(L) = \sum_{i=1}^{2}\left[\left(\frac{r_i^2}{S_i}\right)\left(\lambda_i^2\left(\frac{\bar{M}}{L^2}\right)\right)\right] \equiv \sum_{i=1}^{2}\left[\left(\frac{r_i^2}{B_i}\right)\bar{M}\right] \tag{53}$$

In this instance, $\bar{M}=\int_0^L M^*(x-L)dx=M^*L^2/2$, so that the tip deflection of the beam loaded by the tip moment follows from Eq. (53) as

$$w_{q(x)}(L) = \sum_{i=1}^{2}\left[\left(\frac{r_i^2 L^2}{2B_i}\right)M^*\right] \tag{54}$$

Equation (54) shows that the stiffness of each beam in response to the tip moment M^* can be identified as $K_{TM^*i}=2B_i/L^2$, with a corresponding apportionment factor $r=K_{TM^*1}/(K_{TM^*1}+K_{TM^*2})$, which in turn allows the tip displacement to be written as

$$w_{M^*}(L) = \frac{M^*}{K_{TM^*1} + K_{TM^*2}} = \frac{M^* L^2}{2(B_1 + B_2)} \tag{55}$$

It is important to recognize that the two foregoing results apportion the loads on the coupled Timoshenko beam model in accord with satisfying equilibrium and ensuring compatible displacements, which is precisely the outcome of applying Castigliano's second theorem [10]. It happens that the usual discrete modeling reflected above yields the same result for the discrete tip load and moment.

4 Conclusions

Extensions of the well known Castigliano theorems were developed in the context of modeling the behavior of both discrete coupled systems and various coupled beam sets. It was shown that the proper apportionment of loads among parallel system elements could be determined by minimizing of the displacement of that parallel system as dictated by Castigliano's second theorem. Similarly, it was shown that the correct distribution of the displacements of a set of series elements follows from the minimization of the load in that series system, as defined in Castigliano's first theorem. These extensions also provide a means for apportioning loads in coupled continuous systems, as was shown for the cases of coupled cantilever Timoshenko beams supporting discrete and continuous loads.

Finally, in their various form, Eqs. (44), (49), (52), and (55) make clear that the effective stiffness of this coupled beam model depends on four material properties (E_1, G_1, E_2, G_2) and five geometric properties (I_1, A_1, I_2, A_2, H), albeit encompassed in the shear (S_i) and bending (B_i) stiffnesses. However, it is the pair of dimensionless ratios (λ_i^2, a^2) that are most useful for both analysis estimates and preliminary design for, as shown elsewhere [11], these rather straightforward design guides provide very good agreement with the far more complex exact results.

Acknowledgment

The author is very grateful to his HMC colleague, Emeritus Professor Harry E. Williams, for many helpful conversations during the formalization of this work, and to Harry Dym, Emeritus Professor of Mathematics, Weizmann Institute, for assistance with the proof.

Appendix

A formal derivation of the solution (13) can be found as follows. The basic equations (12) to be solved are written in matrix form as

$$\mathbf{Kr} = \mathbf{v} \tag{A1}$$

Here, $\mathbf{r}$ and $\mathbf{v}$ are column matrices with $n-1$ rows

$$\mathbf{r} \triangleq \begin{bmatrix} r_1 & r_2 & \cdots & r_{n-1} & r_{n-2} \end{bmatrix}^T, \quad \mathbf{v} \triangleq \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \end{bmatrix}^T \tag{A2}$$

and a symmetric stiffness matrix $\mathbf{K}$ of order $n-1$ is defined as

$$\mathbf{K} \triangleq \begin{bmatrix} (k_n/k_1 + 1) & 1 & \cdots & 1 & 1 \\ 1 & (k_n/k_2 + 1) & \cdots & 1 & 1 \\ 1 & 1 & \ddots & 1 & 1 \\ 1 & 1 & \cdots & (k_n/k_{n-2} + 1) & 1 \\ 1 & 1 & \cdots & 1 & (k_n/k_{n-1} + 1) \end{bmatrix} \tag{A3}$$

Note that the quantity 1 can be subtracted from every element of the stiffness matrix (A2), which allows it to be rewritten as the following sum:

$$\mathbf{K} = \mathbf{D} + \mathbf{v}\mathbf{v}^T \tag{A4}$$

where $\mathbf{D}$ is a diagonal matrix of order $n-1$

$$\mathbf{D} \triangleq \begin{bmatrix} k_n/k_1 & 0 & \cdots & 0 & 0 \\ 0 & k_n/k_2 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & k_n/k_{n-2} & 0 \\ 0 & 0 & \cdots & 0 & k_n/k_{n-1} \end{bmatrix} \tag{A5}$$

In view of Eqs. (A2) and (A5), Eq. (A3) can be written as

$$\mathbf{K} = \mathbf{D}(\mathbf{I} + \mathbf{D}^{-1}\mathbf{v}\mathbf{v}^T) \triangleq \mathbf{D}(\mathbf{I} + \mathbf{u}\mathbf{v}^T) \tag{A6}$$

where $\mathbf{I}$ is the unit matrix of order $n-1$ and a new variable $\mathbf{u}$ has been introduced

$$\mathbf{u} \triangleq \mathbf{D}^{-1}\mathbf{v} = (1/k_n)\mathbf{k} \tag{A7}$$

and where in parallel with Eq. (A2) $\mathbf{k}$ is a column matrix with $n-1$ rows

$$\mathbf{k} \triangleq \begin{bmatrix} k_1 & k_2 & \cdots & k_{n-1} & k_{n-2} \end{bmatrix}^T \tag{A8}$$

Now, the matrix $\mathbf{D}$ in Eq. (A6) is invertible since $k_i > 0$ for $i = 1, 2, \ldots, n$. Further, since

$$\mathbf{u}\mathbf{v}^T = \mathbf{D}^{-1}\mathbf{v}\mathbf{v}^T = \frac{1}{k_n}\begin{bmatrix} k_1 & k_1 & \cdots & k_1 & k_1 \\ k_2 & k_2 & \cdots & k_2 & k_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ k_{n-2} & k_{n-2} & \cdots & k_{n-2} & k_{n-2} \\ k_{n-1} & k_{n-1} & \cdots & k_{n-1} & k_{n-1} \end{bmatrix} \tag{A9}$$

it can be shown that (see Exercise 5.13 in Ref. [12])

$$\det(\mathbf{I} + \mathbf{u}\mathbf{v}^T) = \det(1 + \mathbf{v}^T\mathbf{u}) = 1 + \mathbf{v}^T\mathbf{u} \tag{A10}$$

where $\mathbf{v}^T\mathbf{u}$ is a *scalar* that is readily evaluated from Eqs. (A2) and (A7), that is

$$\mathbf{v}^T\mathbf{u} = k_n^{-1}\sum_{i=1}^{n-1} k_i \equiv k_n^{-1}\sum_{i=1}^{n} k_i - 1 \tag{A11}$$

so that the matrix $(\mathbf{I}+\mathbf{u}\mathbf{v}^T)$ is also invertible. Then, by direct calculation it follows that

$$(\mathbf{I} + \mathbf{u}\mathbf{v}^T)^{-1} = \left(\mathbf{I} - \frac{\mathbf{u}\mathbf{v}^T}{1 + \mathbf{v}^T\mathbf{u}}\right) = \left(\frac{(1 + \mathbf{v}^T\mathbf{u})\mathbf{I} - \mathbf{u}\mathbf{v}^T}{1 + \mathbf{v}^T\mathbf{u}}\right) = \left(\frac{\left(k_n^{-1}\sum_{i=1}^{n} k_i\right)\mathbf{I} - \mathbf{u}\mathbf{v}^T}{k_n^{-1}\sum_{i=1}^{n} k_i}\right) \tag{A12}$$

The inverse of the matrix (A6) is then found by substituting Eq. (A12) therein and performing the required inversion, that is

$$\mathbf{K}^{-1} = (\mathbf{I} + \mathbf{u}\mathbf{v}^T)^{-1}\mathbf{D}^{-1} = \left(\frac{\left(k_n^{-1}\sum_{i=1}^{n} k_i\right)\mathbf{I} - \mathbf{u}\mathbf{v}^T}{k_n^{-1}\sum_{i=1}^{n} k_i}\right)\mathbf{D}^{-1} = \mathbf{D}^{-1} - \frac{k_n\mathbf{u}\mathbf{v}^T}{\sum_{i=1}^{n} k_i}\mathbf{D}^{-1} \tag{A13}$$

where the inverse of the diagonal matrix $\mathbf{D}$ is

$$\mathbf{D}^{-1} = \frac{1}{k_n}\begin{bmatrix} k_1 & 0 & \cdots & 0 & 0 \\ 0 & k_2 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & k_{n-2} & 0 \\ 0 & 0 & \cdots & 0 & k_{n-1} \end{bmatrix} \tag{A14}$$

Then the load apportionment factors are

$$\mathbf{r} = \left[\mathbf{D}^{-1} - \frac{k_n\mathbf{u}\mathbf{v}^T}{\sum_{i=1}^{n} k_i}\mathbf{D}^{-1}\right]\mathbf{v} \tag{A15}$$

Equations (A9) and (A14) can then be used to show that

$$\frac{k_n\mathbf{u}\mathbf{v}^T}{\sum_{i=1}^{n} k_i}\mathbf{D}^{-1}\mathbf{v} = \frac{\sum_{i=1}^{n-1} k_i}{k_n\sum_{i=1}^{n} k_i}\mathbf{k} = \frac{\sum_{i=1}^{n} k_i - k_n}{k_n\sum_{i=1}^{n} k_i}\mathbf{k} \tag{A16}$$

And lastly, after substitution of Eqs. (A14) and (A16) into Eq. (A15), the load apportionment factors are found to be exactly those in Eq. (13), that is

$$\mathbf{r} = \left(1 \Big/ \sum_{i=1}^{n} k_i\right)\mathbf{k} \tag{A17}$$

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