Consistent Derivations of Spring Rates for Helical Springs

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The spring rates of a coiled helical spring under an axial force and an axially directed torque are derived by a consistent application of Castigliano's second theorem, and it is shown that the coupling between the two loads may not always be neglected. The spring rate of an extensional spring is derived for the first time through the use of the displacement based principle of minimum total potential energy. The present results are also compared with available derivations of and expressions for the stiffness of a coiled spring. [DOI: 10.1115/1.3125888]

1 Introduction

Modern textbooks on strength or mechanics of materials often use the classic linear spring $F=kx$ to represent or highlight various aspects of behavior, including the bending of beams and the twisting of elastic rods. Interestingly though, while analyses of the extensional spring itself were a staple of earlier texts on the mechanics of materials (i.e., Refs. [1–3]), they are not typically found in current texts on the subject, one exception being that in Ref. [4]. Further, the discussions in the older texts focused primarily on Wahl's strength analysis of the cross section of the curved rods of which the spring is made [5,6], although the spring's deformation was sometimes included in these presentations [1,2].

One exposition of the stiffness of a coiled spring is contained in the 1969 reprinting of Southwell's [7] classic treatise. That work reports correct results obtained with Castigliano's second theorem (CST), but its derivation is in an abbreviated, idiosyncratic form that is not especially useful to students of mechanics or design. The spring is also a topic of interest in books on mechanical design (or mechanical engineering design). Shigley et al. [8] also derived the spring's stiffness (or spring rate or scale) in a manner not unlike Southwell, and while their final, practically useful approximation is also correct, its derivation makes different assumptions. Southwell's derivation includes the complementary energy due to both torsion and a transverse moment, while Shigley, et al., include the complementary energy due to torsion and to transverse shear. Boresi and Schmidt also applied CST in a derivation like Southwell's, but they introduced a (later) minor simplification from the very beginning [4]. It is worth noting that each of these three explications of the spring rate of an extensional spring is based on CST, and that no connection is made to the spring's extensional deformation. That link will be made here via the strain energy based principle of minimum total potential energy (PMTPE), an approach to this problem that does not appear in the literature. Finally, Ref. [9] appears to provide the only analysis of a helical spring under the simultaneous loading of an axial force and an axially directed torque. Wittrick's analysis ignored the effects of shear and axial (along the coil's local axis) deformation and emerges from the present analysis when the appropriate assumptions are made. What also emerges here is the observation that the two loads do produce a coupled deformation pattern that may, in fact, prove significant even at coil pitch angles that are smaller than might be expected from the analysis of the two separate, uncoupled problems.

The present work is motivated by the desire to make available a fully consistent derivation of the state of stress in a coiled helix, including both torsion and transverse components, and to make explicit the several approximations that are generally and appropriately made. These assumptions, both about the coil's dimensions and the spring's helix angle (or pitch) α, underlie the proper application of these derived spring rates to design. Thus, while the general practice of spring design is unlikely to be affected by the availability of a consistent derivation, a new caution emerges, and the long term understanding of future students and practitioners of design can only be enhanced.

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2 Stress Components in a Helical Spring

Using the standard helical coordinates shown in Fig. 1, the distance to any point P on the centerline of the helical coil shown is

$$\mathbf{r}(O/P) = R\mathbf{e}_r + z(\theta)\mathbf{e}_z \tag{1}$$

Then, for the given axial spring force $\mathbf{F}=F\mathbf{e}_z$, a free-body diagram would show that the applied torque produced on a negatively directed face at the point P is

$$\mathbf{M}^F_{\text{applied}} = \mathbf{r}(O/P) \times \mathbf{F} = -FR\mathbf{e}_\theta \tag{2}$$

If a torque T directed along the spring's axis $\mathbf{M}^T_{\text{applied}}=T\mathbf{e}_z$ is added as a load on the spring, the total applied moment is

$$\mathbf{M}_{\text{applied}} = \mathbf{M}^F_{\text{applied}} + \mathbf{M}^T_{\text{applied}} = -FR\mathbf{e}_\theta + T\mathbf{e}_z \tag{3}$$

Conversely, on an internal cross section of the coil a local coordinate system can be identified in terms of the basic helical coordinates as

$$\mathbf{e}_x = \cos\alpha\mathbf{e}_\theta + \sin\alpha\mathbf{e}_z$$

$$\mathbf{e}_y = -\mathbf{e}_r$$

$$\mathbf{e}_{z^*} = -\sin\alpha\mathbf{e}_\theta + \cos\alpha\mathbf{e}_z \tag{4}$$

where $\mathbf{e}_x$ is a unit vector along the coil's tangent line and $\mathbf{e}_{z^*}$ is a unit vector in the plane of the coil wire's cross section as dictated by the vector product $\mathbf{e}_{z^*}=\mathbf{e}_r\times\mathbf{e}_x$, and α is the helix angle of the coil. The moment acting on a slightly inclined (at the helical angle α), positively directed cross section of the coil can then be written as

$$\mathbf{M} = M_x\mathbf{e}_x + M_y\mathbf{e}_y + M_{z^*}\mathbf{e}_{z^*} \tag{5}$$

or, in terms of the original helical coordinate system

$$\mathbf{M} = M_x(\cos\alpha\mathbf{e}_\theta + \sin\alpha\mathbf{e}_z) + M_y\mathbf{e}_y + M_{z^*}(-\sin\alpha\mathbf{e}_\theta + \cos\alpha\mathbf{e}_z) \tag{6}$$

Moment equilibrium is then enforced simply as $\mathbf{M}=\mathbf{M}_{\text{applied}}$, which amounts to simply equating Eq. (3) to Eq. (6). This yields the local components of $\mathbf{M}$ as

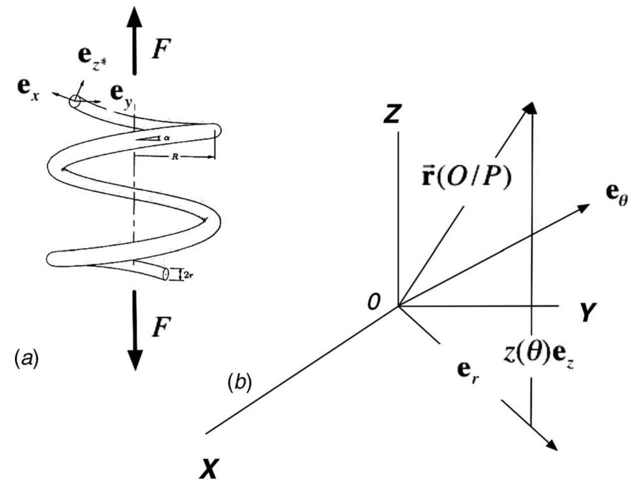

Fig. 1 (*a*) **An axially loaded helical spring and** (*b*) **the coordinate systems for the spring and a face at a point** ***P*** **along the centerline of the coil. The pitch of the spring coil is given by** $p=2\pi R\tan\alpha$.

$$M_x = -FR\cos\alpha + T\sin\alpha$$

$$M_y = 0$$

$$M_{z^*} = FR\sin\alpha + T\cos\alpha \tag{7}$$

Hence, Eq. (5) becomes

$$\mathbf{M} = (-FR\cos\alpha + T\sin\alpha)\mathbf{e}_x + (FR\sin\alpha + T\cos\alpha)\mathbf{e}_{z^*} \tag{8}$$

Similarly, the force acting on the slightly inclined, positively directed cross section of the coil can then be written as

$$\mathbf{F} = N\mathbf{e}_x + F_y\mathbf{e}_y + V\mathbf{e}_{z^*} \tag{9}$$

where N is the (normal) stress resultant directed along the coil's tangent, F_y and V are the radial and transverse (vertical) component of the shear force acting on the plane. Force equilibrium is enforced as $\mathbf{F}=Fe_z$ so that the local components of $\mathbf{F}$ can be found by using the coordinate transformation process just followed:

$$N = F\sin\alpha$$

$$F_y = 0$$

$$V = F\cos\alpha \tag{10}$$

Hence, Eq. (8) becomes

$$\mathbf{F} = F(\sin\alpha\mathbf{e}_x + \cos\alpha\mathbf{e}_{z^*}) \tag{11}$$

Note that all of the force (Eq. (10)) and moment (Eq. (8)) components are constant along the axis of the coil's wire (as distinct from the axis of the spring itself).

3 Deformation of a Helical Spring Via Castigliano's Second Theorem

The deformation of a helical spring is straightforwardly found by applying CST with the stress state defined by Eqs. (6) and (9). Since that state of stress does not vary along the coil, and since the components of Eqs. (6) and (9) reflect torsion along the centerline of the coil, bending normal to the centerline, transverse shear, and axial tension, the complementary energy is [10]

$$U^* = \int_0^{L_c}\left[\frac{M_x^2}{2GJ} + \frac{M_{z^*}^2}{2EI} + \frac{V^2}{2GA} + \frac{N^2}{2EA}\right]dx \tag{12}$$

where x is the local coordinate along the line of centroids of the coil cross sections and L_c is the total length of the coil, that is

$$L_c = n(2\pi R)/\cos\alpha \tag{13}$$

and there are n coils in the spring. The properties of the wire making up the spring coil that appear in Eq. (4) are the elastic modulus E, the shear modulus $G=E/(2(1+\nu))$ with a Poisson ratio ν, the cross-sectional area $A=\pi d^2/4$ with a wire diameter d, the second moment of that area $I=\pi d^4/64$, and the polar moment of that area $J=2I$. Note that all coils are assumed to be active throughout, so that the number of active coils and the total number of coils are one and the same. Further, since none of the various stress resultants varies with the coordinate x along the length of the coil, Eq. (12) can be written as the simpler

$$U^* = \frac{L_c}{2GJ}\left[M_x^2 + \frac{GJ}{EI}M_{z^*}^2 + \frac{J}{A}V^2 + \frac{JG}{EA}N^2\right] \tag{14}$$

With the stress components M_x, M_{z^*}, V, and N defined in Eqs. (7) and (10), the complementary energy (Eq. (14)) can be found to be

$$U^* = \frac{L_cR^2}{2GJ}\left\{\left[\cos^2\alpha + \frac{GJ}{EI}\sin^2\alpha + \frac{J}{AR^2}\cos^2\alpha + \frac{GJ}{EAR^2}\sin^2\alpha\right]F^2 - \left[2\left(1-\frac{GJ}{EI}\right)\sin\alpha\cos\alpha\right]F\left(\frac{T}{R}\right) + \left[\sin^2\alpha + \frac{GJ}{EI}\cos^2\alpha\right]\left(\frac{T}{R}\right)^2\right\} \tag{15}$$

Wittrick's statement of the complementary energy, the only one to attack the coupled problem, can be found in Eq. (15) if the terms J/AR^2 are regarded as small compared with 1 [9]. All of the other analyses are for uncoupled extensional springs ($T=0$), and so the comparisons are with the first bracketed term (that multiplied by F^2) in Eq. (15). Southwell's formulation of the complementary energy contains only the first two terms due to the torsion and the transverse moment [7]. Boresi and Schmidt did the same, except that they approximated the length of the spring as $L_c=n2\pi R$ from the very beginning [4]. The formulation in Ref. [8] contains the first term and the third term with $\cos\alpha$ taken as unity, that is, assuming that the torsion and the shear force act on a vertical face of the coil (as if $\alpha=0$). As will be discussed further below, the second, third, and fourth terms are generally negligible, albeit for somewhat different reasons, so that these differences are unlikely to affect the design and selection of standard extensional springs.

The coil is now assumed to be the usual solid circular wire of diameter d, for which $J=2I$. With $D=2R$, and using both the standard definition of the shear modulus G and the commonly defined spring index $c\equiv D/d$, Eq. (15) can be written as

$$U^* = \left(\frac{n\pi}{8\cos\alpha}\right)\frac{D^3}{GJ}\left\{\left[\left(1+\frac{1}{2c^2}\right)\cos^2\alpha + \left(1+\frac{1}{4c^2}\right)\frac{\sin^2\alpha}{(1+\nu)}\right]F^2 - \left[\frac{2\nu}{(1+\nu)}\sin\alpha\cos\alpha\right]F\left(\frac{T}{R}\right) + \left[\sin^2\alpha + \frac{\cos^2\alpha}{(1+\nu)}\right]\left(\frac{T}{R}\right)^2\right\} \tag{16}$$

Equation (16) can be written in terms of flexibility coefficients [11] as

$$U^* = \tfrac{1}{2}f_{11}F^2 - 2f_{12}FT + \tfrac{1}{2}f_{22}T^2 \tag{17}$$

where the flexibility coefficients are

$$f_{11} = \left(\frac{n\pi D^3}{4GJ\cos\alpha}\right)\left[\left(1+\frac{1}{2c^2}\right)\cos^2\alpha + \left(1+\frac{1}{4c^2}\right)\frac{\sin^2\alpha}{(1+\nu)}\right]$$

$$f_{12} = \left(\frac{\nu n\pi D^2}{2(1+\nu)GJ\cos\alpha}\right)[\sin\alpha\cos\alpha]$$

Table 1 Variation in the extensional stiffness ratio (Eq. (22)) for a spring with spring index $c = D/d$ and pitch angle α that is loaded only by an axial force F (i.e., $T=0$); the Poisson ratio is $\nu=0.30$

c	$k_{11}^0/k_{11}=f_{11}/f_{11}^0$				
	$\alpha=5$ deg	$\alpha=10$ deg	$\alpha=15$ deg	$\alpha=20$ deg	$\alpha=35$ deg
4	1.0312	1.0239	1.0305	1.0298	1.0281
5	1.0199	1.0198	1.0195	1.0191	1.0180
10	1.0050	1.0049	1.0049	1.0048	1.0045
20	1.0013	1.0013	1.0013	1.1102	1.0012

$$f_{22} = \frac{n\pi D}{GJ\cos\alpha}\left[\frac{\cos^2\alpha}{(1+\nu)} + \sin^2\alpha\right] \tag{18}$$

Two response modes are encapsulated in the complementary energy (Eq. (17)): The (net) extension δ_{ext} of the free end of a spring due (largely) to the axially applied force F and the (net) torsion angle θ_{tors} at the free end of the spring due (largely) to the torsion T. These deformations are determined by applying CST to Eq. (17), with the results expressed in terms of the following symmetric flexibility matrix [11]:

$$\begin{Bmatrix}\delta_{\text{ext}}\\ \theta_{\text{tors}}\end{Bmatrix} = \begin{Bmatrix} f_{11} & -f_{12}\\ -f_{12} & f_{22}\end{Bmatrix}\begin{Bmatrix}F\\ T\end{Bmatrix} \tag{19}$$

Several features of Eqs. (18) and (19) are worth noting. First, all other properties being equal, Eq. (18) suggests that the spring's flexibility increases as the number of its coils increases, an assessment that conforms to both experience and intuition. Similarly, the flexibilities increase with the coil's diameter D, albeit to different powers: D^3 for the extensional flexibility f_{11}, D^2 for the coupled flexibility f_{12}, and D^1 for the torsional flexibility f_{22}. Three other behavior indicators warrant further discussion: the appearance of the spring index c in the second and fourth terms of the extensional flexibility f_{11}, the effect of the pitch angle α, and the coupling between the extensional and torsional modes displayed in Eq. (19).

3.1 Spring Index Effects. The spring index c appears only in the influence coefficient f_{11} in terms corresponding to the deformation due to shear and normal force on a nonvertical cross section of the coil's rod, as is evident from an annotated version of that (first) influence coefficient

$$f_{11} = \left(\frac{n\pi D^3}{4GJ\cos\alpha}\right)\left[\begin{array}{c}\underbrace{\cos^2\alpha}_{M_x} + \underbrace{\left(\frac{1}{2c^2}\right)\cos^2\alpha}_{V}\\[2ex] + \underbrace{\frac{\sin^2\alpha}{(1+\nu)}}_{M_{z^*}} + \underbrace{\left(\frac{1}{4c^2}\right)\frac{\sin^2\alpha}{(1+\nu)}}_{N}\end{array}\right] \tag{20}$$

Since the spring index is typically large (i.e., $c \gg 1$) these contributions due to shear and extensional deformation are almost always negligible, which is a result familiar from the beam theory [10]. If the second and fourth terms in f_{11} are neglected, the extensional flexibility becomes

$$f_{11}^0 = \left(\frac{n\pi D^3}{4GJ\cos\alpha}\right)\left[\cos^2\alpha + \frac{\sin^2\alpha}{(1+\nu)}\right] \tag{21}$$

The flexibility coefficients (Eqs. (20) and (21)) can be compared in order to assess the relative importance of the spring index c. Such a comparison would likely have the most meaning when the spring is subjected only to an axial force F (i.e., $T=0$), in which case a ratio of those two influence coefficients is coincident with the inverse ratio of corresponding stiffness coefficients and can be written as

$$\frac{f_{11}}{f_{11}^0} = \frac{\left(1+\frac{1}{2c^2}\right) + \left(1+\frac{1}{4c^2}\right)\frac{\tan^2\alpha}{(1+\nu)}}{1+\frac{\tan^2\alpha}{(1+\nu)}} = \frac{k_{11}^0}{k_{11}} \geq 1 \tag{22}$$

Equation (22) clearly shows that retention of the spring index terms, which are typically neglected or dropped, leads to greater flexibility (or smaller stiffness) because those terms allow shear and extensional deformation that are constrained to vanish when $c=0$. The degree to which that softening occurs is clearly modulated as well by the pitch angle α, as is confirmed by the data displayed in Table 1. These data also suggest that spring index effects can be ignored for most practical applications, as will be done below.

3.2 Pitch Angle Effects. The flexibility coefficients (Eq. (18)) clearly depend on the pitch angle α. Terms including $\sin\alpha$ represent contributions of (1) the applied force F to the coil's axial stress resultant N (on a face inclined at the helix angle α) and to the transverse moment M_z^*, and (2) the applied torque T to the transverse moment M_z^*. When both F and T are applied, and the matrix Eq. (19) is inverted, equilibrium is expressed as

$$\begin{Bmatrix}F\\ T\end{Bmatrix} = \begin{Bmatrix} k_{11}^0 & k_{12}^0\\ k_{12}^0 & k_{22}^0\end{Bmatrix}\begin{Bmatrix}\delta_{\text{ext}}\\ \theta_{\text{tors}}\end{Bmatrix} \tag{23}$$

where the stiffness coefficients (for $c \gg 1$) are

$$k_{11}^0 = \frac{4GJ\cos\alpha}{n\pi D^3}[\cos^2\alpha + (1+\nu)\sin^2\alpha]$$

$$k_{12}^0 = \left(\frac{2\nu GJ\cos\alpha}{n\pi D^2}\right)[\sin\alpha\cos\alpha]$$

$$k_{22}^0 = \frac{EJ\cos\alpha}{2n\pi D}[(1+\nu)\cos^2\alpha + \sin^2\alpha] \tag{24}$$

To more fully explicate the effects of pitch angle, the trigonometric terms are expanded as Taylor series, in which case the stiffness terms become

$$k_{11}^0 \cong \left(\frac{4GJ}{n\pi D^3}\right)(1+\mathrm{O}(\alpha^2))$$

$$k_{12}^0 = \left(\frac{2\nu GJ\alpha}{n\pi D^2}\right)(1+\mathrm{O}(\alpha^2))$$

$$k_{22}^0 = \left(\frac{EJ}{2n\pi D}\right)(1+\mathrm{O}(\alpha^2)) \tag{25}$$

Clearly, for closely packed coil springs where the spring's pitch is very small, the extensional stiffness becomes the "standard" spring rate, that is

Table 2 The variation in the dimensionless constraining torque needed to eliminate the rotation of an extensional spring with pitch angle; the Poisson ratio is $n=0.30$

	$\alpha=5$ deg	$\alpha=10$ deg	$\alpha=15$ deg	$\alpha=10$ deg	$\alpha=15$ deg
$\dfrac{T^*\vert_{\alpha\to 0}}{(FD/2)}$	0.0262	0.0524	0.0785	0.1047	0.1571

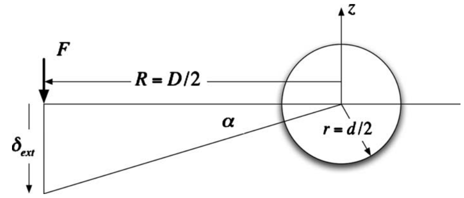

Fig. 2 The dominant part of the extension $\delta_{\text{ext}}=(f_{11}^0+f_{\text{shear}})F \cong f_{11}^0 F$ of a helical extensional spring is due to the twist φ of a section of a rod that results from the downward force F that creates the torque $FD/2$

$$k_{11}^0 \cong \frac{4GJ}{n\pi D^3} \tag{26}$$

This result is exactly that found in Refs. [2,4,7,8,12], and in many other sources. Equations (23)–(25) make explicit the fact that the extensional stiffness of an axial spring is due to the torsional stiffness GJ of the spring coil's cross section. This means that the standard (i.e., close packed) extensional spring can be modeled as a (periodic) set of n straight rods of length $L=2\pi R$ in torsion. Similarly, for such small helical angles, the torsional stiffness becomes

$$k_{22}^0 \cong \frac{EJ}{2n\pi D} \equiv \frac{EI}{n\pi D} \tag{27}$$

Again, this is the standard result found in many places [8,12] and it reflects a similarly "odd" response. That is, while the extensional spring responds in torsion, the torsional spring response is governed by the bending of the coils about an axis normal to the coil wire's centerline.

3.3 Coupling Effects. Perhaps the most intriguing issue is the interpretation of the coupled stiffness k_{12}^0. While it is more than likely that the pitch angle α will be small enough to ignore coupling between extensional and torsional loading and deformation, the foregoing analysis at least suggests that such coupling can occur because k_{12}^0 depends linearly on the pitch angle even in the small angle analysis of Eq. (25). In fact, Eqs. (19) and (23) make clear that even absent an applied torque T, the extensional force will produce a rotation of the spring coil given by

$$\theta_{\text{tors}} = \frac{F}{k_{12}^0} \tag{28}$$

Thus, in principle, there will always be some rotation or torsional motion of an extensional spring subjected only to an axial load. This also means that in order to restrain the spring from such rotation, a constraining torque T^* would have to be developed in an amount

$$T^* = \frac{f_{12}^0}{f_{22}^0}F = \frac{\nu \sin 2\alpha}{\cos^2\alpha + (1+\nu)\sin^2\alpha}\left(\frac{FD}{4}\right) \tag{29}$$

For the typical spring with a small pitch angle, the constraining torque is then,

$$T^*\big|_{\alpha\to 0} \cong \nu\alpha\left(\frac{FD}{2}\right) \tag{30}$$

Thus, the constraining torque T^* required to eliminate rotation of the spring is a fraction of the torque induced in the coil wire by the extensional force (i.e., $FD/2$) that is just $\nu\alpha$. That this fraction is rather small is clearly evident from the numbers presented in Table 2. Of course, this part of the analysis only hints at the complexities involved in examining the end conditions for springs with both small and large pitch angles [6].

4 Spring Rate of an Extensional Spring via Minimum Total Potential

The foregoing derivation, based on CST, depends on identifying the equilibrium state as an input to that theorem. In order to construct a parallel derivation to determine the proper spring constant by using the PMTPE, a proper formulation of the displacement field is needed as input. The axial displacement or extension of a helical spring loaded only by the axial force F can be found by applying Castigliano's theorem to either of Eq. (15) and (16) with the torque T set to zero. With the physical sources from which it derives highlighted, that deflection is

$$\delta_{\text{ext}} = \left(\frac{n\pi D^3}{4GJ\cos\alpha}\right)\left[\begin{array}{c} \underbrace{\cos^2\alpha}_{M_x} + \underbrace{\left(\dfrac{1}{2c^2}\right)\cos^2\alpha}_{V} \\ \\ + \underbrace{\dfrac{\sin^2\alpha}{(1+\nu)}}_{M_{z^*}} + \underbrace{\left(\dfrac{1}{4c^2}\right)\dfrac{\sin^2\alpha}{(1+\nu)}}_{N} \end{array}\right]F \tag{31}$$

For the standard case of closely packed coils $\sin\alpha \cong \alpha \ll 1$, the terms due to M_{z^*} and N can be neglected, and each cosinusoid in Eq. (31) can be replaced by 1. (This also means that the twist θ_{tors} due to F can also be neglected.) Hence,

$$\delta_{\text{ext}} = \left(\frac{n\pi D^3}{4GJ}\right)\left[\underbrace{1}_{M_x} + \underbrace{\left(\frac{1}{2c^2}\right)}_{V}\right]F \equiv f_{11}^0\left(1 + \frac{f_{\text{shear}}}{f_{11}^0}\right)F \tag{32}$$

where, in this instance, f_{11}^0 is the simple inverse of Eq. (26) (and also given by Eq. (21) with $\alpha \cong 0$), and f_{shear} is the flexibility of a simple shear beam of length L_c under a load F

$$f_{\text{shear}} = \frac{L_c}{GA} \tag{33}$$

For a coil of high spring index c, the shear extensional flexibility ratio is very small

$$\frac{f_{\text{shear}}}{f_{11}^0} = \frac{4J}{AD^2} = \frac{1}{2c^2} \ll 1 \tag{34}$$

so the shear deformation can be neglected.

Under these circumstances, the vertical deformation of most significance is just due to the model wherein the standard extensional spring can be viewed as a (periodic) set of n straight rods of length $L=L_c=\pi D$, to each of which a torque is applied along its axis. In this model there is no warping, that is, there is no displacement along the rod's axis and its plane sections remain plane. The helical coil in the spring is assumed to respond similarly, so that the components of displacement parallel to the y and z axes of a cross section (see Fig. 2) are, respectively,

$$v(x,y,z) = -\alpha_t zx$$

$$w(x,y,z) = \alpha_t yx \tag{35}$$

where α_t is the rate of twist of the plane section. (Note that most texts use an unadorned α to denote the twist rate of a circular rod, but that variable was already used for the pitch angle of the coil, which is also widely used notation). The shear strains corresponding to the displacements (Eq. (31)) are then easily found to be

$$\gamma_{yx} = -\alpha_t zx$$

$$\gamma_{zx} = \alpha_t yx \tag{36}$$

The strain energy U due to torsion in n of such rods can then be calculated from [10]

$$U = n\tfrac{1}{2}G\int_0^{L_c}\int_A [\gamma_{yx}^2 + \gamma_{zx}^2]dAdx \tag{37}$$

which in view of Eq. (36) can be integrated and cast in terms of the twist rate α_t as

$$U = n\tfrac{1}{2}GJL_c\alpha_t^2 \tag{38}$$

Now, for a straight circular rod under torsion, there is no net vertical deflection of the rod because the centroids of each of the sections do not move. However, imagine that the torque T that causes torsion of the coil is applied through a vertical force F that is applied along a diametrical line whose length equals that of the coil radius $R=D/2$ (see Fig. 2). As the vertical force F moves downward a distance δ_c while applying a torque to the end of the rod (at $x=L_c \cong 2\pi R$), the coil cross section experiences a twist angle φ that is related to the twist rate α_t by

$$\varphi = \alpha_t x \tag{39}$$

The displacement δ_c, the vertical displacement of the point (L_c, $-d/2, 0$), and the twist angle φ are related by similar triangles

$$\tan\varphi = \frac{\delta_c}{D/2} \cong \frac{w(L_c, -d/2, 0)}{d/2} \tag{40}$$

Then, in view of Eq. (40) and the second of Eq. (35), the displacement δ_c for small twist angles φ can be cast in terms of the twist rate α_t as

$$\delta_c \cong (D/2)\varphi = (D/2)L_c\alpha_t = (\pi D^2/2)\alpha_t \tag{41}$$

Equation (41) is true for one coil. If there are n of such coils in the spring, the total vertical displacement of the force F (which is also the extension of the coiled spring) that causes the torque is then

$$\delta_{\text{ext}} = n\delta_c = (n\pi D^2/2)\alpha_t \tag{42}$$

Hence the strain energy for n coils is found by substituting Eq. (42) into Eq. (38) which results to

$$U = \frac{2GJ\delta_{\text{ext}}^2}{n\pi D^3} \tag{43}$$

Then the total potential energy Π is found by adding the potential of the applied axial load F to the strain energy (Eq. (43))

$$\Pi = \frac{2GJ\delta_{\text{ext}}^2}{n\pi D^3} - F\delta_{\text{ext}} \tag{44}$$

In accord with the PMTPE, the vanishing of the first variation in the total potential energy of Eq. (44) then yields the equation of equilibrium of the extensional spring, expressed in terms of the displacement δ_{ext}:

$$\left(\frac{4GJ}{n\pi D^3}\right)\delta_{\text{ext}} = F \tag{45}$$

The spring rate for the standard extensional spring is readily apparent in Eq. (45).

5 Conclusions

The extensional and torsional stiffness of a standard helical spring under combined axial forces F and torques T were derived from basic principles. The first derivation showed that a complete analysis based on Castigliano's second theorem confirmed well known results while illustrating the minor (and generally inconsequential) inconsistencies of previous derivations. Further, it also showed that some coupling between the extensional and rotational responses occurs even at small pitch angles. It is worth noting, however, that the coupling exposed above is limited by the constraints of linear elasticity, in contrast with the changes in spring geometry and stiffness due to the spring strain that emerge in a nonlinear analysis [5,6]. The torsional stiffness of a helical spring also emerged from this derivation, albeit with a better founded equilibrium condition than that typically portrayed in mechanical design textbooks. The second derivation of the extensional stiffness showed for the first time that the same result could also be obtained using the principle of minimum total potential energy.

The results clearly show that the extensional stiffness varies linearly with the torsional stiffness of the rod cross section and inversely with the number of spring coils. The stiffness also softens as the helix angle (or the pitch) of the coil increases, so that loosely packed springs are less stiff than closely packed springs (for which the effect of the pitch angle may be neglected). It is also interesting that the first of Eq. (18) suggests that a thicker rod also tends to increase the spring's extensional flexibility, likely because it allows local shear and extensional deformation that are constrained to vanish in an analysis that does not admit them.

Finally, it is worth noting that the foregoing analyses do not include the effects of varying end conditions. End conditions are likely best modeled using finite element analysis (FEA) customized for the specific situation in which the spring is used, once a final design of the spring body was done with the formulations developed above.

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