Estimating Fundamental Frequencies of Tall Buildings

Clive L. Dym, P.E., F.ASCE[1]; and Harry E. Williams[2]

Abstract: Empirical estimates of the fundamental frequency of tall buildings vary inversely with their height, a dependency not exhibited by the various familiar models of beam behavior. This paper examines and explains this apparent discrepancy by analyzing the consequences of using two models to estimate such natural frequencies: A two-beam model that couples the bending of a classical cantilever to that of a shear beam by imposing a displacement constraint; and a Timoshenko beam in which the Euler–Bernoulli beam model is extended by adding a shear-displacement term to the classical bending deflection. A comparison of the two beam models suggests that the Timoshenko model is appropriate for describing the behavior of shear-wall buildings, while the coupled two-beam model is appropriate for shear-wall–frame (e.g., tube-and-core) buildings, and that the coupled-beam model comes much closer to replicating the parametric dependence of building frequency on height.

DOI: XXXX

CE Database subject headings: Buildings, high-rise; Estimation; Frequency.

Introduction

There is little doubt about the value of the ability to rapidly estimate the natural frequency of high-rise buildings—both for design and pedagogical purposes. Indeed, it is often stated that, to first order, a tall building excited by wind or seismic forces responds as an elementary cantilever beam (Taranath 1988). According to elementary beam theory, the classic Euler–Bernoulli beam (EBB) result for the lowest resonance frequency of a uniform cantilever is given as [e.g., (Dym and Shames 1973)]

$$\omega_{\text{EBB}} \cong (1.875)^2 \sqrt{\frac{EI}{\rho A H^4}} \sim \frac{1}{H^2} \tag{1}$$

where H=height of the building being modeled. Thus, the natural or fundamental frequency is inversely proportional to the square of that building's height.

At the same time, the literature features several instances of empirically derived formulas that allow a user to estimate the lowest natural frequency of a tall building as a function of H. For example, Newmark and Hall (1981) suggested that the fundamental frequency n_0 for braced-steel frames and reinforced-concrete shear-wall buildings depends on the reciprocal of N, the number of stories in the building, which is essentially proportional to the building's height. Ellis (1980), after measuring the frequencies of 163 buildings, recommended a similar formula that varied as the reciprocal of the building's height, H. Smith and Coull (1991) note a "widely used" formula that is particularly applicable to reinforced-concrete shear-wall buildings and braced-steel frames (Newmark and Hall 1981), and that showed a direct variation with the square root of D, the building depth, with the same inverse dependence on H. The result of Newmark and Hall (1981) is also said to be applicable to shear-wall construction (Lee et al. 2000). Goel and Chopra (1997) suggested that for steel moment-resisting frame structures the fundamental frequency should vary as $H^{-0.8}$, and for reinforced-concrete moment-resisting frames as $H^{-0.9}$. Finally, Goel and Chopra (1998) also suggested that for concrete shear-wall buildings the fundamental frequency should vary as $1/H$.

All of the results cited show the same (or nearly the same) dependence, namely, frequency $\sim 1/H$, which is in sharp contrast to the beam-model predictions. Thus, it is natural to ask whether a tall building's frequency *can* be estimated from a beam-theory calculation, or if the commonly used empirical formulas indicate a different behavior. A related question is whether a higher-order beam theory, such as the well-known Timoshenko beam model, provides a resolution of this apparent conflict.

This paper answers these questions by considering two different models of building-as-beam behavior, and for both models the squares of the fundamental frequencies reflect the sum of a "bending" term and a "shear term" that can be expressed in one of two equivalent forms that display their dependence on building height explicitly

$$\omega^2 \sim \left(\frac{c_L}{H}\right)^2 \left(\frac{I_b}{AH^2}\right)(C_{\text{bending}} + \alpha^2 C_{\text{shear}}) \tag{2a}$$

and

$$\omega^2 \sim \left(\frac{c_S}{H}\right)^2 \left(\frac{C_{\text{bending}}}{\alpha^2} + C_{\text{shear}}\right) \tag{2b}$$

Eqs. (2a) and (2b) are cast in terms of the longitudinal and shear wave speeds, respectively, $c_L^2 = E_b/\rho$ and $c_S^2 = kG_s/\rho$, as well as two dimensionless parameters, C_{bending} and C_{shear}, which depend only on mode shape. Both expressions of each of the two models also share a common, critical, dimensionless parameter, α (Miranda and Taghavi 2005)

[1]Fletcher Jones Professor of Engineering Design, Dept. of Engineering, Harvey Mudd College, Claremont, CA 91711-5990. E-mail: dym@hmc.edu

[2]Emeritus Professor of Engineering, Dept. of Engineering, Harvey Mudd College, Claremont, CA 91711-5990.

Note. Associate Editor: Abhinav Gupta. Discussion open until March 1, 2008. Separate discussions must be submitted for individual papers. To extend the closing date by one month, a written request must be filed with the ASCE Managing Editor. The manuscript for this technical note was submitted for review and possible publication on February 13, 2006; approved on April 4, 2007. This technical note is part of the ***Journal of Structural Engineering***, Vol. 133, No. 10, October 1, 2007. ©ASCE, ISSN 0733-9445/2007/10-1–XXXX/$25.00.

$$\alpha^2 \equiv \frac{kG_sA_sH^2}{E_bI_b} \tag{3}$$

The parameter α is proportional to the ratio of the beam's (or building's) *shear stiffness*, $\sim kG_sA_s/H$, to its *bending stiffness*, E_bI_b/H^3. Thus, the parameter α reflects the interaction of both geometric and material properties.

The two beam models considered herein are:

- A coupled two-beam (CTB) model consisting of an elementary EBB, required to have the same displacement of the shear beam (SB) to which it is tied (Heidebrecht and Smith 1973; Rutenberg 1975; Balendra 1984; Basu et al. 1982, 1984; Miranda 1999; Miranda and Reyes 2002; Miranda and Taghavi 2005). The CTB model is shown to approach the EBB model for $\alpha^2 \ll 1$ and it behaves as an SB for $\alpha^2 \gg 1$, in which case the fundamental frequency varies as $1/H$. The CTB model is appropriate for modeling shear wall-frame (e.g., tube-and-core) construction.
- A Timoshenko-beam (TB) model that accounts for shear deformation by adding it to the bending deflection (Dym and Shames 1973). The TB model is shown to approach the SB model for $\alpha^2 \ll 1$ and it behaves as an EBB for $\alpha^2 \gg 1$, in which case the fundamental frequency varies as $1/H^2$. The TB model can be used to model shear-wall construction.

It is worth noting that the difference in the limiting behaviors of the CTB and TB models can be characterized in terms of the fact that the TB model reflects a *series* coupling of the beam's bending and shear stiffnesses, while the CTB model couples the bending and shear stiffnesses in *parallel*. Finally, it is shown that the $P-\Delta$ effect due to the building's (or beam's) self-weight is negligible for both models.

Frequency–Height Dependence in Coupled Two-Beam Models

Consider modeling a building as a vertically oriented cantilever beam in which the cross section is comprised of two beams connected in parallel. For example, consider a symmetric external tube and a symmetric internal core, made of different materials, and connected by axially rigid, massless transverse elements. The external tube is modeled as an elementary beam with the usual bending stiffness, E_bI_b, and a centerline transverse displacement of $w_b(x,t)$. The internal core has standard shear stiffness, G_sA_s, whose transverse displacement is $w_s(x,t)$. The rigid transverse connectors ensure that the (beam) bending and (core) shear displacements are equal, i.e.

$$w_b(x,t) = w_s(x,t) \equiv w(x,t) \tag{4}$$

where $w(x,t)$=common transverse displacement of the coupled two-beam model of the tube and core. (The present analysis can be easily extended to include cross sections that vary over the building's height without changing the basic result presented below.)

The governing equation for the coupled beam can be found by applying Hamilton's principle to the appropriate total Lagrangian, L_T. The total Lagrangian can be obtained simply by summing the two kinetic energy terms and subtracting from that sum the total potential energy for the elementary beam and the core (Dym and Shames 1973). The kinetic energies for the elementary beam, T_b, and for the shear beam, T_s, are

$$T_b + T_s = \frac{\rho_bA_b}{2}\int_0^H \left(\frac{\partial w(x,t)}{\partial t}\right)^2 dx + \frac{\rho_cA_c}{2}\int_0^H \left(\frac{\partial w(x,t)}{\partial t}\right)^2 dx \tag{5}$$

and the strain energies for the EBB, U_b, and for the shear beam, U_s, are

$$U_b + U_s = \frac{E_bI_b}{2}\int_0^H \left(\frac{\partial^2 w(x,t)}{\partial x^2}\right)^2 dx + \frac{G_sA_s}{2}\int_0^H \left(\frac{\partial w(x,t)}{\partial x}\right)^2 dx \tag{6}$$

Finally, the vertical (in this instance) load due to the (vertically oriented) beam's own weight per unit length, q_0, can be developed in terms of the moment it produces in the beam at a location x from the fixed support at the base. For cantilever boundary conditions, this result can also be expressed as an equivalent potential of the axial load q_0 (Dym and Shames 1973)

$$V_q = -\frac{q_0H}{2}\int_0^H \left(1-\frac{x}{H}\right)\left(\frac{\partial w(x,t)}{\partial x}\right)^2 dx \tag{7}$$

It is worth noting that the form of Eq. (7) is identical in form—save for a different load—to the potential energy of an axial load P applied along the axis of an elastic column (Dym 2002). This term produces here the second-order effect of the gravity loads, known as the $P-\Delta$ effect (Smith and Coull 1991). Then the total potential energy for the CTB model is, finally

$$\begin{aligned} L_T = {} & \frac{\rho_bA_b}{2}\int_0^H \left(\frac{\partial w(x,t)}{\partial t}\right)^2 dx + \frac{\rho_sA_s}{2}\int_0^H \left(\frac{\partial w(x,t)}{\partial t}\right)^2 dx \\ & - \left[\frac{E_bI_b}{2}\int_0^H \left(\frac{\partial^2 w(x,t)}{\partial x^2}\right)^2 dx + \frac{G_sA_s}{2}\int_0^H \left(\frac{\partial w(x,t)}{\partial x}\right)^2 dx \right. \\ & \left. - \frac{q_0H}{2}\int_0^H \left(1-\frac{x}{H}\right)\left(\frac{\partial w(x,t)}{\partial x}\right)^2 dx\right] \end{aligned} \tag{8}$$

The Lagrangian (8) can be applied together with Hamilton's principle to derive the equation of motion for the coupled composite beam, along with the appropriate boundary conditions. However, because the present interest is solely in finding a useful estimate of that beam's fundamental frequency, the Lagrangian (8) can be used to develop a Rayleigh quotient for the beam by assuming the usual separable solution

$$w(x,t) = W(\xi)\cos \omega t \tag{9}$$

where $\xi = x/H$=dimensionless axial coordinate. The substitution of the solution (9) into the Lagrangian (8) then yields an equation for the fundamental frequency of the CTB model. If a *total* specific mass ρA is defined as $\rho A = \rho_bA_b + \rho_sA_s$, that Rayleigh quotient appears as

$$\omega_{\mathrm{CTB}}^2 = \frac{\dfrac{E_b I_b}{2H^3}\displaystyle\int_0^1 [W''(\xi)]^2 d\xi - \frac{q_0}{2}\int_0^1 (1-\xi)[W'(\xi)]^2 d\xi + \frac{G_s A_s}{2H}\int_0^1 [W'(\xi)]^2 d\xi}{\dfrac{\rho A H}{2}\displaystyle\int_0^1 [W(\xi)]^2 d\xi} d\xi \tag{10}$$

The Rayleigh quotient (10) can be written as the sum of three separate terms

$$\omega_{\mathrm{CTB}}^2 = \left(\frac{c_S}{H}\right)^2 \left[\frac{C_{\mathrm{bending}}^{\mathrm{CTB}}}{\alpha^2} + C_{\mathrm{bending}}^{\mathrm{CTB}} - \left(\frac{(q_0 H/A)}{kG_s}\right) C_{P-\Delta}^{\mathrm{CTB}}\right] \tag{11}$$

where the three dimensionless coefficients in Eq. (11) are

$$C_{\mathrm{bending}}^{\mathrm{CTB}} = \frac{\displaystyle\int_0^1 [W''(\xi)]^2 d\xi}{\displaystyle\int_0^1 [W(\xi)]^2 d\xi}, \quad C_{\mathrm{shear}}^{\mathrm{CTB}} = \frac{\displaystyle\int_0^1 [W'(\xi)]^2 d\xi}{\displaystyle\int_0^1 [W(\xi)]^2 d\xi}$$

$$C_{\mathrm{bending}}^{\mathrm{CTB}} = \frac{\displaystyle\int_0^1 (1-\xi)[W'(\xi)]^2 d\xi}{\displaystyle\int_0^1 [W(\xi)]^2 d\xi} \tag{12}$$

Each of the coefficients defined in Eq. (12) is a simple number, of order unity, that depends solely on the mode shape $W(\xi)$ and not on either the mode shape's amplitude or frequency.

The decomposition in Eq. (11) states that the resonance frequency is proportional to the sum of three terms: the first incorporates bending and is inversely proportional to H^2, the second term incorporates shear deformation and is inversely proportional to H, and the third term incorporates $P-\Delta$ effects that are, in fact, negligible. This follows from the fact that the coefficient that precedes the third coefficient is effectively the ratio of the normal stress produced by the building's entire weight, $q_0 H/A$, to the building's shear modulus, kG_s. Since stresses are typically smaller than one-thousandth of a modulus, the $P-\Delta$ term can be neglected. Then Eq. (11) reduces to

$$\omega_{\mathrm{CTB}}^2 = \left(\frac{c_S}{H}\right)^2 \left(\frac{C_{\mathrm{bending}}^{\mathrm{CTB}}}{\alpha^2} + C_{\mathrm{shear}}^{\mathrm{CTB}}\right) \tag{13}$$

which was anticipated in Eq. (2*b*). Then, depending on the relative magnitude of the various parameters of the coupled two-beam model, either form of dependency might influence the frequency more: The CTB model approaches the EBB model for $\alpha^2 \ll 1$ and the SB model for $\alpha^2 \gg 1$, in which case the fundamental frequency varies as $1/H$.

It is important to keep in mind that the results presented in Eqs. (11)–(13) were derived based on the assumption that the bending and shear deflections were one and the same. This suggests an analogy to the connection of the beam's (or building's) mass to two springs acting in parallel, one representing the bending response, and the other, the shear response.

Frequency–Height Dependence in Timoshenko Beams

Consider once again a vertically oriented, cantilever beam with a uniform cross section that is symmetric through the thickness. (Again, the analysis presented now is also easily extended to include nonuniform cross sections, without changing the basic result presented below.) The beam model presented now is a TB model that incorporates both shear deformation and rotatory inertia (Dym and Shames 1973; Rahgozar et al. 2004), although the rotatory-inertia effect is unimportant here because it is not a significant factor for tall slender buildings. Let $w(x,t)$ denote the total transverse displacement of the beam's centerline, $\psi(x,t)$ the bending rotation of line elements originally normal to the beam's centerline, $\beta(x,t)$ the rotation of the centerline normal due to shear, and x the beam's axial coordinate. The formulation of the displacement field for this problem begins with the assumption that the total slope of the beam is the sum of a component due to the bending rotation $\psi(x,t)$ and the shear rotation $\beta(x,t)$ (Dym and Shames 1973)

$$\frac{\partial w(x,t)}{\partial x} = \psi(x,t) + \beta(x,t) \tag{14}$$

Displacement and strain fields that correspond to Eq. (14) are easily determined (Dym and Shames 1973), although it is worth noting that the axial strain is linear through the beam's thickness, and the shear-strain constant. These are unrealistic assumptions whose effects will be compensated below with the introduction of a shear constant.

The Rayleigh quotient for the TB corresponding to Eq. (14) can be developed in a process much like that presented for the coupled two-beam model, and it produces a result that is very similar in structure and appearance to that displayed in Eq. (11). Including the $P-\Delta$ effect but ignoring rotatory inertia, the TB result is

$$\omega_{\mathrm{TB}}^2 = \left(\frac{c_S}{H}\right)^2 \left[\frac{C_{\mathrm{bending}}^{\mathrm{TB}}}{\alpha^2} + C_{\mathrm{shear}}^{\mathrm{TB}} - \left(\frac{(q_0 H/A)}{kG_s}\right) C_{P-\Delta}^{\mathrm{TB}}\right] \tag{15}$$

The coefficients displayed in Eq. (15) are, for the TB model

$$C_{\text{bending}}^{\text{TB}} = \frac{\int_0^1 [\Psi'(\xi)]^2 d\xi}{\int_0^1 [W(\xi)]^2 d\xi}$$

$$C_{\text{shear}}^{\text{TB}} = \frac{\int_0^1 [W'(\xi) - \Psi(\xi)]^2 d\xi}{\int_0^1 [W(\xi)]^2 d\xi}$$

$$C_{P-\Delta}^{\text{TB}} = \frac{\int_0^1 (1-\xi)[W'(\xi)]^2 d\xi}{\int_0^1 [W(\xi)]^2 d\xi} \tag{16}$$

It is worth noting that the Rayleigh quotient corresponding to the bending vibrations of a self-loaded EBB can be found from Eq. (16) simply by setting $\Psi(\xi) = W'(\xi)$. Unfortunately, no such transition exists by which the coupled two-beam model can be obtained from the Timoshenko quotient. This is almost certainly due to the fact that the TB model is based on the assumption that the total slope of the deflected beam is the sum of the bending and shear rotations. This suggests that the TB model is analogous to a bending spring and a shear spring connected in series to the beam's (or building's) mass.

Choosing an Elementary Continuum Model

Each of the two beam models presented above have been suggested as appropriate estimates of the dynamic response of tall buildings: The coupled or composite beam in Miranda and Taghavi (2005) and Taghavi and Miranda (2005), and the Timoshenko model in Rahgozar et al. (2004). Both seem to offer similar results in the sense that each can be expressed in the form of Eqs. (2*a*) and (2*b*), in spite of the fact that there underlying displacement assumptions are decidedly different. The coupled-beam model represents a parallel formulation, while the Timoshenko model suggests a series formulation. Further, neither model conforms with the empirical results described in the Introduction. Is either model adequate or appropriate?

In fact, while the CTB and TB models appear to have similar forms [viz, Eqs. (11) and (12), and (15) and (16)], they are not at all the same, with the difference traceable to the distinction drawn between the parallel and series formulations. In the case of Eq. (12) the coefficients are independent of the mode amplitude, because there is only a single degree of freedom in the CTB model since the transverse displacement is clearly the same for both beams.

For the CTB model results of Eq. (13), with the coefficients defined by Eq. (12), it is clear that for very small values of α (or large values of the slenderness ratio) the bending term in the brackets of Eq. (20) dominates, and the limiting CTB frequency becomes that of an EBB, i.e.

$$\omega_{\text{CTB}}^2(\alpha^2 \ll 1) \sim \left(\frac{c_S}{H}\right)^2 \frac{C_{\text{bending}}^{\text{CTB}}}{\alpha^2} = \frac{E_b I_b}{\rho A H^4} \frac{\int_0^1 [W''(\xi)]^2 d\xi}{\int_0^1 [W''(\xi)]^2 d\xi} \tag{17}$$

For the first mode of a cantilever (and without the axial self-load), Eq. (17) reduces to the classical EBB result (Dym and Shames 1973)

$$\omega_{\text{CTB}}^2(\alpha^2 \ll 1) \Rightarrow \omega_{\text{EBB}}^2 = (1.875)^4 \left(\frac{E_b I_b}{\rho A H^4}\right) \tag{18}$$

It is clear from both Eqs. (17) and (18) that neither result conforms to the empirical data outlined in the Introduction. Thus, it is also clear that the first mode of a tall building cannot be estimated as if it were a classical EBB.

For very large values of α (or small values of the slenderness ratio) the second term in the brackets of Eq. (13) can be neglected, and the limiting two-beam frequency becomes that of a SB, i.e.

$$\omega_{\text{CTB}}^2(\alpha^2 \gg 1) \sim \left(\frac{c_S}{H}\right)^2 C_{\text{shear}}^{\text{CTB}} = \frac{G_s A_s}{\rho A H^2} \frac{\int_0^1 [W'(\xi)]^2 d\xi}{\int_0^1 [W(\xi)]^2 d\xi} \tag{19}$$

For the first mode of a cantilever SB, for which $W(\xi) \sim \sin(\pi\xi/2)$, Eq. (19) reduces to the classical (SB) result

$$\omega_{\text{CTB}}^2(\alpha^2 \gg 1) \Rightarrow \omega_{\text{SB}}^2 = \left(\frac{\pi}{2}\right)^2 \left(\frac{G}{\rho H^2}\right) \tag{20}$$

Here it can be seen that Eqs. (19) and (20) both conform to the empirical data outlined in the Introduction, that is, $\omega \sim 1/H$. Thus, it seems clear that the first mode of a tall building can be estimated as if it were a SB.

A further confirmation of this conclusion is that the result (20) can be used to calculate the speed of a shear wave in the beam (building) as a function of frequency. In particular, Eq. (20) can be solved for the shear wave speed expressed as the modulus-to-density ratio, with the frequency also recast as $\omega = 2\pi n$

$$\sqrt{\frac{G_s}{\rho}} = \left(\frac{2}{\pi}\right)\omega_{\text{CTB}} H = 4 n_{\text{CTB}} H \tag{21}$$

Now, using the empirical results presented in the Introduction, the shear wave speeds can be explicitly calculated. For example, using Ellis' estimate, as described earlier, yields a shear speed of 184 m/s. Further, using the Newmark–Hall estimate, with the additional assumption that each story is 3.5 m high, yields a shear speed of 140 m/s. These values are consistent with previously reported estimates of the shear wave speed (Safak 1999).

Would an entirely similar analysis of the TB produce the same results, or at least some that are similar? The short answer is, "No." The reason is as mentioned earlier: There are two degrees of freedom in that model, representing separately (if summed) the bending deformation and the shear deformation. This inevitably makes Eqs. (15) and (16) dependent on (modal) amplitudes, which thus removes the possibility of a straightforward analysis, such as that done for the coupled two-beam model, and that the frequency behavior observed in gatherings of empirical data cannot be replicated.

Conclusions

In conclusion, it appears that the CTB model seems the better—if not the "best"—model for estimating the frequencies of shear-wall-frame buildings because it provides predictions that are consistent with the observed data. Further, the kinematic behavior assumed in the model also seems to fit the physics, e.g., compare tube-and-core construction with the parallel nature of the two-beam model in which transverse displacements due to bending and to shear are identical. The TB model, on the other hand, cannot exhibit the correct behavior, and the classical elementary EBB results also do not display the correct dependence of frequency on beam (building) height.

References

Balendra, T. (1984). "Free vibration of a shear wall-frame building on an elastic foundation." *J. Sound Vib.*, 96(4), 437–446.

Basu, A. K., Nagpal, A. K., and Kaul, S. (1984). "Charts for seismic design of frame-wall systems." *J. Struct. Eng.*, 110(1), 31–46.

Basu, A. K., Nagpal, A. K., and Nagar, A. K. (1982). "Dynamic characteristics of frame-wall systems." *ASCE J. Struct. Div.*, 108(6), 1201–1218.

Dym, C. L. (2002). *Stability theory and its applications to structural mechanics*, Dover, Mineola, N.Y.

Dym, C. L., and Shames, I. H. (1973). *Solid mechanics: A variational approach*, McGraw-Hill, N.Y.

Ellis, B. R. (1980). "An assessment of the accuracy of predicting the fundamental natural frequencies of buildings and the implications concerning the dynamic analysis of structures." *Proc., Inst. Struct. Engineers*, 69(2), 763–776.

Goel, R. K., and Chopra, A. K. (1997). "Period formulas for moment resisting frame buildings." *J. Struct. Eng.*, 123(11), 1454–1461.

Goel, R. K., and Chopra, A. K. (1998). "Period formulas for concrete shear wall buildings." *J. Struct. Eng.*, 124(4), 426–433.

Heidebrecht, A. C., and Smith, S. B. (1973). "Approximate analysis of tall wall-frame structures." *ASCE J. Struct. Div.*, 99(2), 199–221.

Lee, L.-H., Chang, K.-K., and Chun, Y.-S. (2000). "Experimental formula for the fundamental period of RC buildings with shear-wall dominant systems." *Struct. Des. Tall Build.*, 9, 295–307.

Miranda, E. (1999). "Approximate seismic lateral deformation demands in multistory buildings." *J. Struct. Eng.*, 125(4), 417–425.

Miranda, E., and Reyes, C. J. (2002). "Approximate lateral drift demands in multistory buildings with nonuniform stiffness." *J. Struct. Eng.*, 128(7), 840–849.

Miranda, E., and Taghavi, S. (2005). "Approximate floor acceleration demands in multistory buildings. I: Formulation." *J. Struct. Eng.*, 131(2), 203–211.

Newmark, N. M., and Hall, W. J. (1981). *Earthquake spectra and design*, Earthquake Engineering Research Institute, Berkeley, Calif.

Smith, B. S., and Coull, A. (1991). *Tall building structures: Analysis and design*, Wiley, New York.

Rahgozar, R., Safari, H., and Kaviani, P. (2004). "Free vibration of tall buildings using Timoshenko beams with variable cross-section." *Structures Under Shock and Impact VIII*, N. Jones and C. A. Brebbia eds., WIT Press, U.K.

Rutenberg, A. (1975). "Approximate natural frequencies of shear walls." *Earthquake Eng. Struct. Dyn.*, 4(1), 95–100.

Safak, E. (1999). "Wave propagation formulation of seismic response of multistory buildings." *J. Struct. Eng.*, 125(4), 426–437.

Taghavi, S., and Miranda, E.(2005). "Approximate floor acceleration demands in multistory buildings. II: Applications." *J. Struct. Eng.*, 131(2), 212–220.

Taranath, B. S. (1988). *Structural analysis and design of tall buildings*, McGraw-Hill, New York.