Using Rotations to Obtain Compatibility Equations for Statically Indeterminate Trusses

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Abstract: This paper presents a new way of determining the compatibility conditions for statically indeterminate trusses. It is based on equations relating the angular displacements (rotations) of truss elements to the corresponding extensions obtained by requiring the displacements of the truss structure to be compatible, i.e., single valued. This new angular displacement method is likely to be most advantageous for planar trusses, but less likely so for three-dimensional structures.

DOI: 10.1061/(ASCE)0733-9445(2005)131:2(353)

CE Database subject headings: Elastic deformation; Equilibrium; Trusses; Static structural analysis; Rotation.

Introduction

Problems involving statically indeterminate trusses or frameworks are usually solved using methods such as the direct stiffness method (e.g., Martin 1966) or Castigliano's second theorem (e.g., Dym 1997), even at the introductory level. However, if enough compatibility equations can be found to augment the equations of equilibrium, then the problem is rendered determinate and thus made easier to solve. Such compatibility equations, obtained from kinematical considerations, are relations between the extensions (or strains) that permit the solution to exist. They are analogous to the equations of compatibility in the theory of elasticity that ensure that the displacements of the structure are continuous. The simplest example of a compatibility equation is the requirement that the sum of the individual bar extensions must vanish in an axially loaded, two-bar system (AB,BC) of pin-ended bars pinned at ends A and C and loaded at the intermediate node, B. This requirement results from the observation that the sum of the relative displacements of the ends of bars AB and BC must vanish. Clearly, once the additional compatibility equations are obtained, the extensions can be related to the bar forces through appropriate constitutive relations to completely fix the solution.

Pellegrino (1990) suggested an alternative approach, relying heavily on a careful application of linear algebra, for solving statically indeterminate problems. Here, the compatibility equations are not used explicitly, but enter implicitly when evaluating the contribution to the bar force solution attributable to the redundants, that is, to the part of the solution that lies in the nullspace of the equilibrium matrix $[A]$.

In this context, the compatibility equations can be identified after the basis vectors $\{x\}$ for the nullspace of $[A]$ are found, where the equilibrium equations take the form

$$[A]\{t\}=\{p\} \tag{1}$$

where $\{t\}$ and $\{p\}$=bar force vector and nodal load vector, respectively. If the vectors $\{x\}$ are the basis of the nullspace, that is, if they satisfy $[A]\{x\}=\{0\}$, it follows from the associated kinematic equations relating the displacement vector to the extension vector $\{e\}$ that the extension vector can have no component in the nullspace, that is, $\{x\}^T\{e\}=\{0\}$. These conditions are the required compatibility conditions. Pellegrino and Calladine (1986) noted that s, the number of compatibility equations, should be $s=b-r$, where b is the number of bars and r is the rank of the equilibrium matrix $[A]$. This number s is also equal to the number of redundant bar forces or states of self-stress in the structure.

The angular displacement method proposed here is an attempt to attach additional physical significance to the solution process. It is particularly well suited to planar structures. In principle, it is also applicable to three-dimensional problems, however, for such problems the work needed to apply the angular displacement method is comparable to that of the linear algebra method and so its use cannot be justified. After developing the angular displacement method, we present applications to several truss structures. These applications show that successful application of the angular displacement method depends on applying common sense geometrical reasoning (e.g., writing equations in terms of the rotation of the bar common to two contiguous triangles, breaking down or decomposing more complex indeterminate frameworks into sets of contiguous triangle pairs, and so on). This approach could be very useful pedagogically (were this level of material still taught in elementary and midlevel courses!).

Angular Displacement Method

This method is based on the observation that, for statically indeterminate trusses, the number of equations relating the angular displacements or rotations of the individual bars to the bar extensions is greater than the number of bar rotations. Thus, if there is

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Note. Associate Editor: Marc I. Hoit. Discussion open until July 1, 2005. Separate discussions must be submitted for individual papers. To extend the closing date by one month, a written request must be filed with the ASCE Managing Editor. The manuscript for this paper was submitted for review and possible publication on August 20, 2003; approved on April 27, 2004. This paper is part of the ***Journal of Structural Engineering***, Vol. 131, No. 2, February 1, 2005. ©ASCE, ISSN 0733-9445/2005/2-353–358/$25.00.

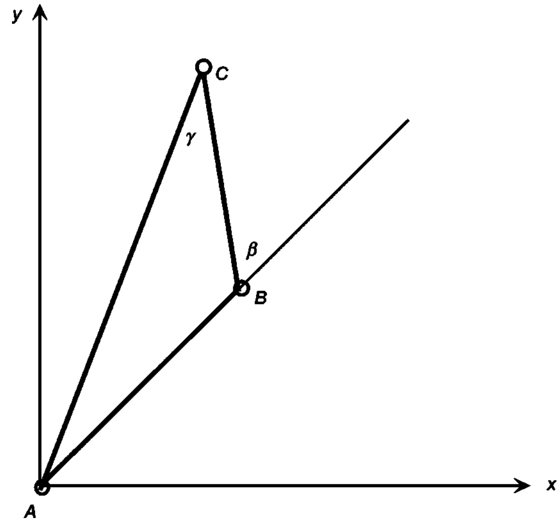

Fig. 1. Basic triangular structure

to be a unique solution for the rotations, a particular relation between the individual bar extensions must exist. We now formulate such equations for a typical planar truss and show that this is indeed the case.

Consider a bar segment defined by the vector $\mathbf{r}(B/A)$ of length ℓ_{AB} that connects node A, located at $\mathbf{r}(A/O)$, to node B, located at $\mathbf{r}(B/O)$, and points toward B from A along the unit vector $\mathbf{e}(B/A)$. Due to deformation, node A moves to $\mathbf{r}(A/O)+\boldsymbol{\delta}(A)$, and node B moves to $\mathbf{r}(B/O)+\boldsymbol{\delta}(B)$, where $\boldsymbol{\delta}(A)$ and $\boldsymbol{\delta}(B)$ are the displacements of the nodes A and B, respectively. Clearly, the difference in position is the deformed line segment $\mathbf{r}^*(B/A)$:

$$\mathbf{r}^*(B/A) = \mathbf{r}(B/A) + \boldsymbol{\delta}(B) - \boldsymbol{\delta}(A) \tag{2}$$

We now define the relative displacement of node B with respect to node A as having a tangential component, $\ell_{AB}\varepsilon_{AB}$, and a normal component, $\ell_{AB}\alpha_{AB}$:

$$\boldsymbol{\delta}(B) - \boldsymbol{\delta}(A) = \ell_{AB}(\varepsilon_{AB}\mathbf{e}(B/A) + \alpha_{AB}\mathbf{n}(B/A)) \tag{3}$$

where $\mathbf{n}(B/A)=\mathbf{e}_z\times\mathbf{e}(B/A)=$unit normal to the line segment that lies in the plane of the structure defined by the unit normal $\mathbf{e}_z$. The parameter ε_{AB} can be identified as the engineering strain in the bar AB, while the parameter α_{AB} reduces to the angle of rotation of the bar AB for small angles of rotation. Then, we can write the deformed line segment as

$$\mathbf{r}^*(B/A) = (1 + \varepsilon_{AB} + \alpha_{AB}\mathbf{e}_z\times)\mathbf{r}(B/A) \tag{4}$$

Note that a positive rotation angle corresponds to a counterclockwise rotation about the z axis.

Consider now the triangular system of bars shown in Fig. 1 connecting nodes A, B, and C. Using the notation introduced for the segment AB, we observe that the undeformed identity

$$\mathbf{r}(C/A) = \mathbf{r}(B/A) + \mathbf{r}(C/B) \tag{5}$$

becomes

$$\mathbf{r}^*(C/A) = \mathbf{r}^*(B/A) + \mathbf{r}^*(C/B) \tag{6}$$

in the deformed state. Hence, on substituting into Eq. (6) the form given by Eq. (4) for the deformed segment AB and corresponding forms for bars BC and AC, it follows that the strain and rotation components must satisfy

$$[\varepsilon_{AB} - \varepsilon_{AC} + (\alpha_{AB} - \alpha_{AC})\mathbf{e}_z\times]\mathbf{r}(B/A) + [\varepsilon_{BC} - \varepsilon_{AC} + (\alpha_{BC} - \alpha_{AC})\mathbf{e}_z\times]\mathbf{r}(C/B) = 0 \tag{7}$$

where we have eliminated $\mathbf{r}(C/A)$ using the undeformed identify of Eq. (5). Eq. (7) can be regarded as defining the relative rotations of the segments AB and BC with respect to the rotation of the segment AC, expressed in terms of the relative strains of the segments AB, BC, and AC.

Since Eq. (7) is a vector equation with two components, it can be solved for the relative bar rotations by taking components along the nonparallel segments AB and BC. With the angles β, γ identified in Fig. 1, and introducing scalar extensions $e_{AB}\equiv\ell_{AB}\varepsilon_{AB}$, etc., we find the results:

$$(\alpha_{BC} - \alpha_{AC})\ell_{BC}\sin\beta = e_{AB} + e_{BC}\cos\beta - e_{AC}\cos(\beta-\gamma) \tag{8a}$$

$$-(\alpha_{AB} - \alpha_{AC})\ell_{AB}\sin\beta = e_{BC} + e_{AB}\cos\beta - e_{AC}\cos\gamma \tag{8b}$$

Eq. (6) essentially requires that the sum of the relative displacements taken around the planar loop ABC vanish. For this basic structure, the rotation α_{AC} can be considered arbitrary and the relative rotations $\alpha_{BC}-\alpha_{AC}$ and $\alpha_{AB}-\alpha_{AC}$ are determined uniquely in terms of the extensions by Eqs. (8*a*) and (8*b*). As will be seen, the number of equations of this type that are available depends on the number of independent loop equations. For statically indeterminate problems, the number of equations relating the rotations (or relative rotations) is greater than the number of rotations (or relative rotations). Hence, for a solution for the rotations to exist, there must exist a particular requirement or requirements—the required compatibility equations—between the bar extensions.

Eqs. (8*a*) and (8*b*) are the compatibility conditions that result from requiring that the displacements form a compatible set. They represent the essence of the *angular displacement method*: They provide additional equations that, added to the equations of bar equilibrium, enable a complete solution to be obtained for the bar forces in an indeterminate truss problem.

First let us note that Eqs. (8*a*) and (8*b*) can also be derived by applying the law of cosines to the deformed triangle with vertices A^*, B^*, and C^*. The sides of the deformed triangle are $\ell_{AB}+e_{AB}$, etc. Further, because of the sign convention used for the rotation angles, the deformed interior angles at the vertices A^* and C^* are, respectively,

$$(\beta - \gamma - (\alpha_{AB} - \alpha_{AC}) \quad \text{and} \quad (\gamma + \alpha_{BC} - \alpha_{AC})$$

Charlton (1982) apparently overlooked the importance of including the effects of angular displacements when he derived Levy's (1886) results using Pythagoras' Theorem. Charlton justified the use of Pythagoras' Theorem, which applies only to right triangles, by noting that "…the elongations are small and insufficient to change the essential geometry…" Though this observation is the basis for writing equilibrium in the undeformed state, it cannot be used to calculate strain components with a formalism that is restricted to right angles. Charlton could have used the Law of Cosines to account for the expected changes in bar orientation. Fortunately, the effects of bar rotation cancel for the examples noted in Charlton (1982) and the correct result was obtained. It should also be noted that the results presented in Examples 1–3 below can also be obtained with the Law of Cosines, although it is not recommended for Example 4 as it becomes too cumbersome.

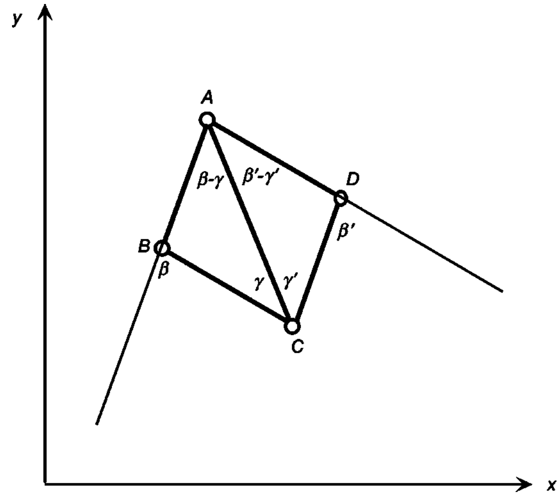

Fig. 2. Contiguous triangles

As a final note in this section, let us use the procedure noted above to record expressions for the bar rotations for the pair of contiguous triangles shown in Fig. 2. We apply essentially the same process as above, but including the additional observation that, in the deformed state, we have

$$\mathbf{r}^*(C/A) = \mathbf{r}^*(D/A) + \mathbf{r}^*(C/D) \tag{9}$$

The corresponding rotation angles for this triangle that complement Eqs. (8*a*) and (8*b*) are given by

$$(\alpha_{AD} - \alpha_{AC})\ell_{AD} \sin \beta' = e_{CD} + e_{AD} \cos \beta' - e_{AC} \cos \gamma' \tag{10a}$$

$$-(\alpha_{CD} - \alpha_{AC})\ell_{(CD)} \sin \beta' = e_{AD} + e_{CD} \cos \beta' - e_{AC} \cos(\beta' - \gamma') \tag{10b}$$

Eqs. (8*b*) and (10*a*) allow us to relate the rotation of bar *AB* to bar *AD*. In what follows, we will present some example problems that illustrate the usefulness of Eqs. (8) and (10) for developing required compatibility equations.

Example 1. Indeterminate Three-Bar Truss Structure

We illustrate the use of rotations to determine compatibility equations for the three-bar structure shown in Fig. 3. The structure is statically indeterminate as there are only two equations of nodal equilibrium available to determine the three-bar forces. There are also three-bar rotations and, by applying Eqs. (8) and (10) to the two contiguous triangles, we can obtain four equations for those three rotations. Clearly, in this case, there must exist only one compatibility equation. In general, we need not determine all of the bar rotations to generate compatibility equations. As we will see, the advantage of this method depends on developing strategies to compute as few rotations as possible.

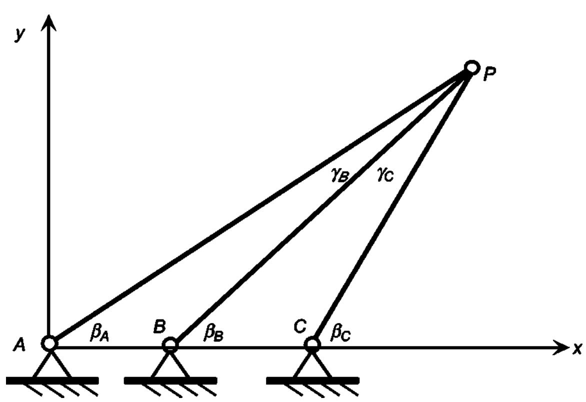

Fig. 3. Indeterminate three-bar structure

Since this structure is essentially composed of two contiguous triangles, we can determine the rotation in the common bar *BP* by using equations appropriate to the triangles *ABP* and *BCP*. Then, having found two expressions for the rotation of the common bar, the required compatibility equation is obtained by equating these two expressions. As a preliminary, we identify the vertices given by *ABC* in Fig. 2 with those given by *PAB* in Fig. 3, and those given by *ADC* in Fig. 2 with those given by *PCB* in Fig. 3. Further, we identify the following angle correspondences:

$$\beta = \pi - \beta_A, \quad \beta' = \beta_C, \quad \gamma = \pi - \beta_B, \quad \gamma' = \beta_B \tag{11}$$

Hence, it follows from Eq. (8*a*) that

$$\alpha_{BP}\ell_{AB} \sin \beta_A = -e_{AP} + e_{BP} \cos(\beta_B - \beta_A) \tag{12}$$

and from Eq. (10*b*) that

$$\alpha_{BP}\ell_{BC} \sin \beta_C = e_{CP} - e_{BP} \cos(\beta_C - \beta_B) \tag{13}$$

Note that we have introduced the observation that the extension and rotation associated with the segments *AB* and *BC* vanish. Thus, upon equating these two expressions and using the law of sines to express ℓ_{AB} and ℓ_{BC} in terms of ℓ_{BP}, we obtain the (single) required compatibility equation:

$$e_{CP} \sin \gamma_B + e_{AP} \sin \gamma_C = e_{BP} \sin(\gamma_B + \gamma_C) \tag{14}$$

Example 2. Five-Bar Indeterminate Truss

As a further illustration of strategies needed to employ this method efficiently, consider the five-bar "tower" structure shown in Fig. 4. This structure is again statically indeterminate as there are four nodal equations of equilibrium available to determine the five-bar forces. Note also that we can obtain three pairs of equations by applying Eqs. (8) and (10) to the triangles *ABP*, *BCP*,

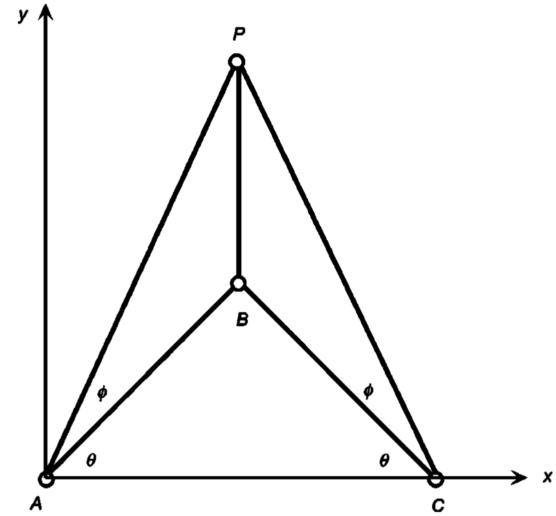

Fig. 4. Indeterminate five-bar truss

and ABC. Hence, observing that the distance AC is fixed, we can obtain six equations for the five rotations, which means that there is one equation of compatibility.

Our strategy for this problem is to first use our results for contiguous triangles to obtain the rotation of bar AB in terms of the rotation of bar BP from an analysis of the triangle ABP, and then to obtain the rotation of bar BC in terms of the rotation of bar BP from an analysis of the triangle BCP. By eliminating the common bar rotation of BP between these two rotations, we find a single relation in terms of the rotations of bars AB and BC.

Once this equation is obtained, we use Eqs. (8a) and (8b) to obtain two additional relations for the bar rotations of AB and BC, using the observation that the (hypothetical) bar AC is rigid. Thus, with the bar rotations of AB and BC now known, we can substitute this result into the previous relation to obtain the required compatibility equation.

In order to carry out the first step, we relate Fig. 4 to Fig. 2 by identifying the vertices ABC in Fig. 2 with the vertices PAB in Fig. 4, and the vertices ADC in Fig. 2 with vertices PCB in Fig. 4. Further, we identify the angle correspondence given by

$$\beta = \beta' = \pi - \phi, \quad \gamma = \gamma' = \pi/2 + \theta \tag{15}$$

Hence, it follows from Eq. (8a) that

$$(\alpha_{AB} - \alpha_{BP})\ell \sin \phi = e_{AP} - e_{AB} \cos \phi - e_{BP} \sin(\theta + \phi) \tag{16}$$

and from Eq. (10b) that

$$(\alpha_{BP} - \alpha_{BC})\ell \sin \phi = e_{CP} - e_{BC} \cos \phi - e_{BP} \sin(\theta + \phi) \tag{17}$$

Upon eliminating the rotation of BP between Eqs. (16) and (17), we obtain

$$\begin{aligned}(\alpha_{AB} - \alpha_{BC})\ell \sin \phi = e_{AP} + e_{CP} - (e_{AB} + e_{BC})\cos \phi \\ - 2e_{BP} \sin(\theta + \phi)\end{aligned} \tag{18}$$

The second step is carried out by, once again, relating the notation of Fig. 4 to that of Fig. 2. Here, we identify the vertices ADC of Fig. 2 with the vertices ABC of Fig. 4, and the angle correspondence given by

$$\beta' = 2\theta, \quad \gamma' = \theta \tag{19}$$

Hence, noting that bar AC is rigid, it follows from Eqs. (10a) and (10b) that

$$\begin{aligned}\alpha_{AB}\ell \sin 2\theta &= e_{BC} + e_{AB} \cos 2\theta \\ -\alpha_{BC}\ell \sin 2\theta &= e_{AB} + e_{BC} \cos 2\theta\end{aligned} \tag{20}$$

Now, with the rotations of bars AB and BC known, we return to our previous result and obtain, upon substituting for the rotations of bars AB and BC, that the extensions must satisfy the compatibility relation that

$$(e_{AP} + e_{CP})\sin \theta = (e_{AB} + e_{BC} + 2e_{BP} \sin \theta)\sin(\theta + \phi) \tag{21}$$

Note that in the limit that $\theta = 0$, we recover the compatibility equation for the axial system of bars found earlier.

Example 3. Indeterminate Six-Bar Truss

In this section, we analyze a structure that is unconstrained by boundary conditions. As it can be oriented arbitrarily in space, its bar rotations can only be determined to within an arbitrary constant. Shown in Fig. 5, this six-bar indeterminate truss is an example used quite often to illustrate the use of Castigliano's second theorem (e.g., Dym 1997). Note that the two diagonal bars AC and BD are not connected at their apparent point of intersection. This truss is statically indeterminate as the eight nodal equations of equilibrium are constrained by overall force and moment equilibrium to yield only five independent equations for the determination of the six-bar forces. It may also be observed that there are three independent triangles, say ABC, ADC, and ABD, that can be analyzed using Eqs. (8) and (10) to yield six equations in the six-bar rotations. However, as one of the bar rotations must be left arbitrary, the six equations in the five remaining bar rotations leads to one equation of compatibility. It should be noted that requiring the displacements to be continuous around the triangle BCD is not an independent requirement because it follows from the corresponding equations appropriate to the three triangles ABC, ADC, and ABD.

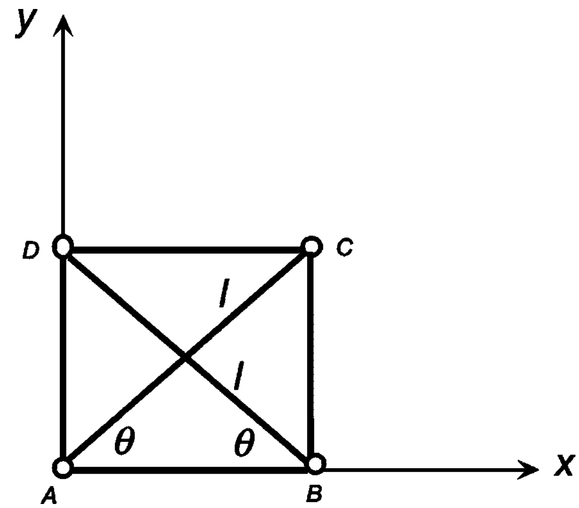

Fig. 5. Indeterminate six-bar truss

Our strategy for this problem is to, first, use our results for contiguous triangles to obtain the rotation of bar AB in terms of the rotation of bar AC from an analysis of the triangle ABC, and, second, obtain the rotation of bar AD in terms of the rotation of bar AC from an analysis of the triangle ADC.

Once these rotations are determined, we can analyze the triangle ABD to obtain the rotations of bars AD and AB in terms of the rotation of bar BD. We can regard the expression for the rotation of bar AD as essentially defining the rotation of bar BD in terms of the rotation of bar AC as

$$\alpha_{BD} - \alpha_{AC} = (\alpha_{BD} - \alpha_{AD}) + (\alpha_{AD} - \alpha_{AC}) \tag{22}$$

and the right-hand side is known. Finally, as the remaining expression for the rotation of bar AB can be similarly expressed in terms of known results, i.e.,

$$\alpha_{AB} - \alpha_{BD} = (\alpha_{AB} - \alpha_{AC}) - (\alpha_{BD} - \alpha_{AC}) \tag{23}$$

we obtain the required compatibility equation.

Proceeding as before, we find from an analysis of the triangle ABC, using Eq. (10b), and noting that $\beta = \pi/2$, $\gamma = \pi/2 - \theta$:

$$(\alpha_{AC} - \alpha_{AB})\ell \cos \theta = e_{BC} - e_{AC} \sin \theta \tag{24}$$

Similarly, we find from an analysis of the triangle ADC, using Eq. (10a), and noting that $\beta' = \pi/2$ and $\gamma' = \theta$:

$$(\alpha_{AD} - \alpha_{AC})\ell \sin \theta = e_{CD} - e_{AC} \cos \theta \tag{25}$$

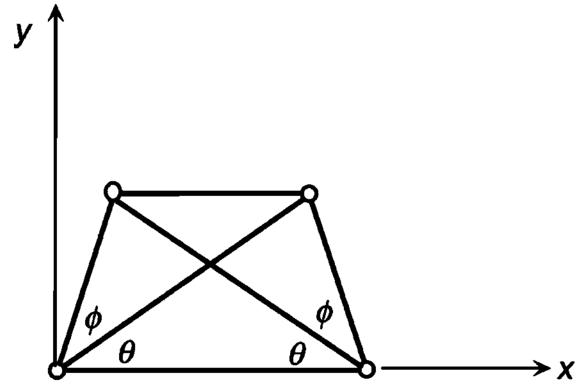

Fig. 6. Trapezoidal indeterminate six-bar truss

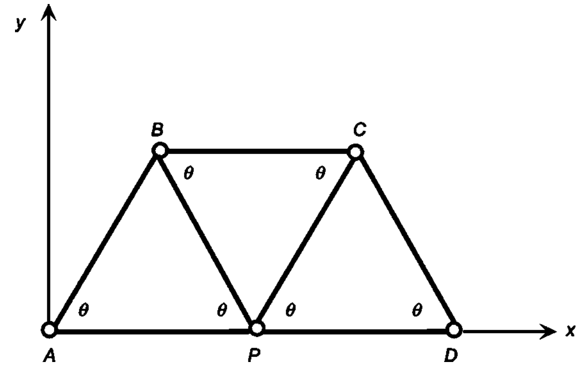

Fig. 8. Basic bar pattern behind the truss of Fig. 7

We obtain the remaining expressions by identifying the vertices of *ABC* in Fig. 2 with those of *DAB* in Fig. 5. Further, we identify the angle correspondence as $\beta=\pi/2$ and $\gamma=\theta$. Hence, it follows from Eqs. (10*a*) and (10*b*) that

$$(\alpha_{AB}-\alpha_{BD})\ell\cos\theta=e_{AD}-e_{BD}\sin\theta$$

$$(\alpha_{BD}-\alpha_{AD})\ell\sin\theta=e_{AB}-e_{BD}\cos\theta \tag{26}$$

Thus, we find

$$(\alpha_{BD}-\alpha_{AC})\ell\sin\theta=e_{AB}+e_{CD}-(e_{AC}+e_{BD})\cos\theta \tag{27}$$

and the requisite compatibility equation becomes:

$$e_{AC}+e_{BD}=(e_{AB}+e_{CD})\cos\theta+(e_{BC}+e_{AD})\sin\theta \tag{28}$$

The result [Eq. (28)] agrees with that presented by Charlton (1982) as being due to Levy (1886).

We could proceed in the same manner to obtain the compatibility equation for other six-sided configurations with arbitrary geometry. Here we present, without proof, the compatibility equation for the trapezoidal six-bar truss shown in Fig. 6

$$(e_{AC}+e_{BD})\sin(\theta+\phi)=(e_{BC}+e_{AD})\sin\theta+(e_{AB}+e_{CD})\sin(2\theta+\phi) \tag{29}$$

Note that Eq. (29) reduces to Eq. (28) for the case that $\phi+\theta=\pi/2$.

Example 4. Repetitive Truss Structure

As a final example, let us consider the truss shown in Fig. 7. Note that it consists of repetitive triangular segments. We choose all of the inclined bars to be of length ℓ. The truss is mounted on rollers at the nodes *E* and *F*, thus rendering it statically indeterminate: There are ten independent nodal equations of equilibrium for the determination of the eleven bar forces. We also observe that this truss consists of five triangular segments, which suggests that there are ten equations similar to Eqs. (8) and (10) for the determination of the eleven bar rotations. However, in order to fix the structure in space, we require that nodes *E* and *F* not move vertically. These two additional equations lead to twelve equations that can be used to determine the eleven bar rotations. This suggests that, once again, there is only one equation of compatibility.

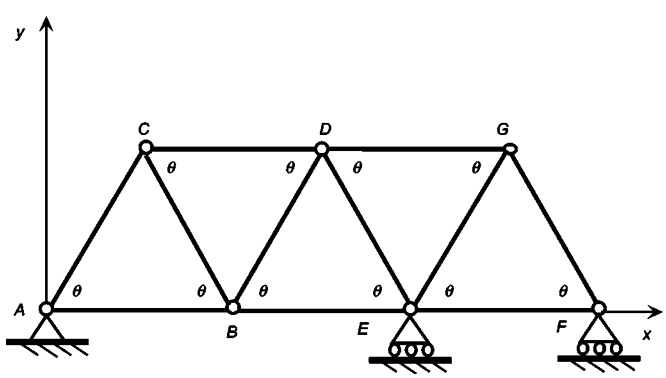

Fig. 7. Repetitive truss structure

As a preliminary to the solution of this problem, we observe that the structure can be thought of as consisting of a pattern of three triangles surrounding the node *B*, and a similar pattern of triangles surrounding the node *E*. This suggests that we analyze the pattern of three triangles shown in Fig. 8 and then develop a strategy for using this type of structure in general.

We are guided in this process by a further observation from our analyses of two contiguous triangles, namely, we could relate the rotation of bars in contiguous triangles to the rotation of the common bar. We can continue this process by noting that we can relate the rotation of bars in any number of contiguous triangles.

In particular, we find from an analysis of the triangle *APB* of Fig. 8 that

$$(\alpha_{AP}-\alpha_{BP})\ell\sin 2\theta=e_{AB}-e_{AP}\cos\theta+e_{BP}\cos 2\theta \tag{30}$$

and, from an analysis of the triangle *BPC* that

$$(\alpha_{BP}-\alpha_{CP})\ell\sin\theta=e_{BC}-(e_{BP}+e_{CP})\cos\theta \tag{31}$$

and, from an analysis of the triangle *CPD* that

$$(\alpha_{CP}-\alpha_{DP})\ell\sin 2\theta=e_{CD}+e_{CP}\cos 2\theta-e_{DP}\cos\theta \tag{32}$$

Thus, on essentially adding these expressions, we find

$$(\alpha_{AP}-\alpha_{DP})\ell\sin 2\theta=(e_{AB}+2e_{BC}\cos\theta+e_{CD})-(e_{AP}+e_{DP})\cos\theta \\ -(e_{BP}+e_{CP}) \tag{33}$$

We observe that the rotation of bar *AP* (to the left of the vertex *P*) minus the rotation of bar *PD* (to the right of the vertex *P*) is the sum of an expression representing the rim bars times minus an expression representing the bottom spoke bars times $\cos\theta$ minus an expression representing the inner spoke bars. This observation allows us to use this result to continue the pattern along a repetitive bar truss.

Returning now to the truss of Fig. 7, we can use the basic bar result of Eq. (33) to write the following for the nodes *B* and *E*:

$$(\alpha_{AB}-\alpha_{BE})\ell \sin 2\theta = e_{AC}+2e_{CD}\cos\theta + e_{DE}-(e_{AB}+e_{BE})\cos\theta - e_{BC}-e_{BD} \quad (34a)$$

and

$$(\alpha_{BE}-\alpha_{EF})\ell \sin 2\theta = e_{BD}+2e_{DG}\cos\theta + e_{GF}-(e_{BE}+e_{EF})\cos\theta - e_{DE}-e_{EG} \quad (34b)$$

As was suggested earlier, we can obtain two additional relations for these bottom bar rotations by invoking fixity conditions at the nodes E and F. Clearly, $\alpha_{EF}=0$ as bar EF can only move laterally, and as the node E can only move laterally:

$$\alpha_{AB}+\alpha_{BE}=0 \quad (35)$$

Now, with four equations in the three bottom bar rotations, it readily follows that the required compatibility equation is

$$2(e_{CD}+2e_{DG})\cos\theta-(e_{AB}+3e_{BE}+2e_{EF})\cos\theta+e_{AC}-e_{BC}+e_{BD} - e_{DE}-2e_{EG}+2e_{GF}=0 \quad (36)$$

It should be noted that changing the fixity condition at node F to a pinned end leads to an additional requirement that the sum of the extensions of the bottom bars must vanish. This follows from an equation similar to Eqs. (8*a*) and (8*b*) involving the three bars AB, BE, and EF, by enforcing the requirement that the node F cannot move in the x direction.

Discussion and Conclusions

It has been shown in the above examples that an additional equation of compatibility can often be determined so that the bar forces in a statically indeterminate structure can be evaluated without also having to evaluate the nodal displacements. Eqs. (14), (21), (28), (29), and (36) are examples of such compatibility equations. Once the compatibility equation for a statically indeterminate structure is determined, it is necessary only to introduce an appropriate constitutive relation to obtain the necessary additional equation to fix the solution.

The method of rotations requires one at the outset to develop some physical insight into the problem. This may be considered an advantage in introductory courses and is in contrast to linear algebra methods which can be reduced to simply vector and matrix manipulations. The first step in this method is to determine the number of independent displacement compatibility equations, i.e., the number of loops around which the displacements must be continuous. This step essentially establishes the number of rotation–extension equations governing the problem. Generally, there are more equations (say n) than rotations (say m) for a statically indeterminate problem. In matrix form, the left-hand side of the equations relating the rotations to the extensions takes the form of an $n\times m$ coefficient matrix operating on the vector of rotations. The coefficient matrix is generally of rank m, so that a solution for the rotations can be found provided that there exist $n-m$ equations of compatibility. Example 3 is a special case in that the structure is free to rotate about the z axis. In the present context, the six equations in six rotations should be regarded as six equations in five relative rotations so that, again, one should expect one equation of compatibility.

In the second step, it is necessary to identify which of the resulting rotation–extension equations need be solved so that the compatibility equation can be determined efficiently. If this step cannot be carried out and the full set of equations need to be row reduced to produce the required compatibility equation, the method loses its advantage. All of the examples shown above illustrate this process. Generally, only a small number of the total number of rotations in the problems entered into the calculation.

As can be seen from the examples presented above, the angular displacement or rotation method is well suited to efficiently determining compatibility equations for planar basically triangular structures, i.e., trusses. The restriction to planar structures is based on the observation that implementation of this method for a three-dimensional structure would generally require as much work as the linear algebra method and hence offer no advantage. The restriction to structures composed of triangular substructures follows, as it is only for such structures that efficient strategies can be determined to obtain the required compatibility equations without computing essentially all bar rotations. There is the additional attractive feature of triangular structures that they are sufficiently rigid so as to prevent mechanisms from appearing. Not only are four-sided substructures unlikely to be analyzed by such efficient strategies as noted above, but they can introduce mechanisms that cast the problem into one of being both statically and kinematically indeterminate. Such problems are clearly beyond the scope of this paper (and likely, most introductory courses).

Notation

The following symbols are used in the paper:

- $\mathbf{e}(B/A)$ = unit vector;
- e_{AB} = scalar extensions of the strain ε_{AB};
- ℓ_{AB} = length of vector node A to node B before deformation;
- $\mathbf{n}(B/A)$ = unit normal to the line segment ℓ_{AB};
- $\{p\}$ = nodal load vector;
- $\mathbf{r}(B/A)$ = vector connecting node A to node B before deformation;
- $\mathbf{r}^*(B/A)$ = vector connecting node A to node B after deformation;
- $\{t\}$ = bar force vector;
- α_{AB} = rotation angle of truss element AB during deformation;
- $\boldsymbol{\delta}(A)$, $\boldsymbol{\delta}(B)$ = displacement vectors of nodes A, B; and
- ε_{AB} = engineering strain of truss element between nodes A and B.

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